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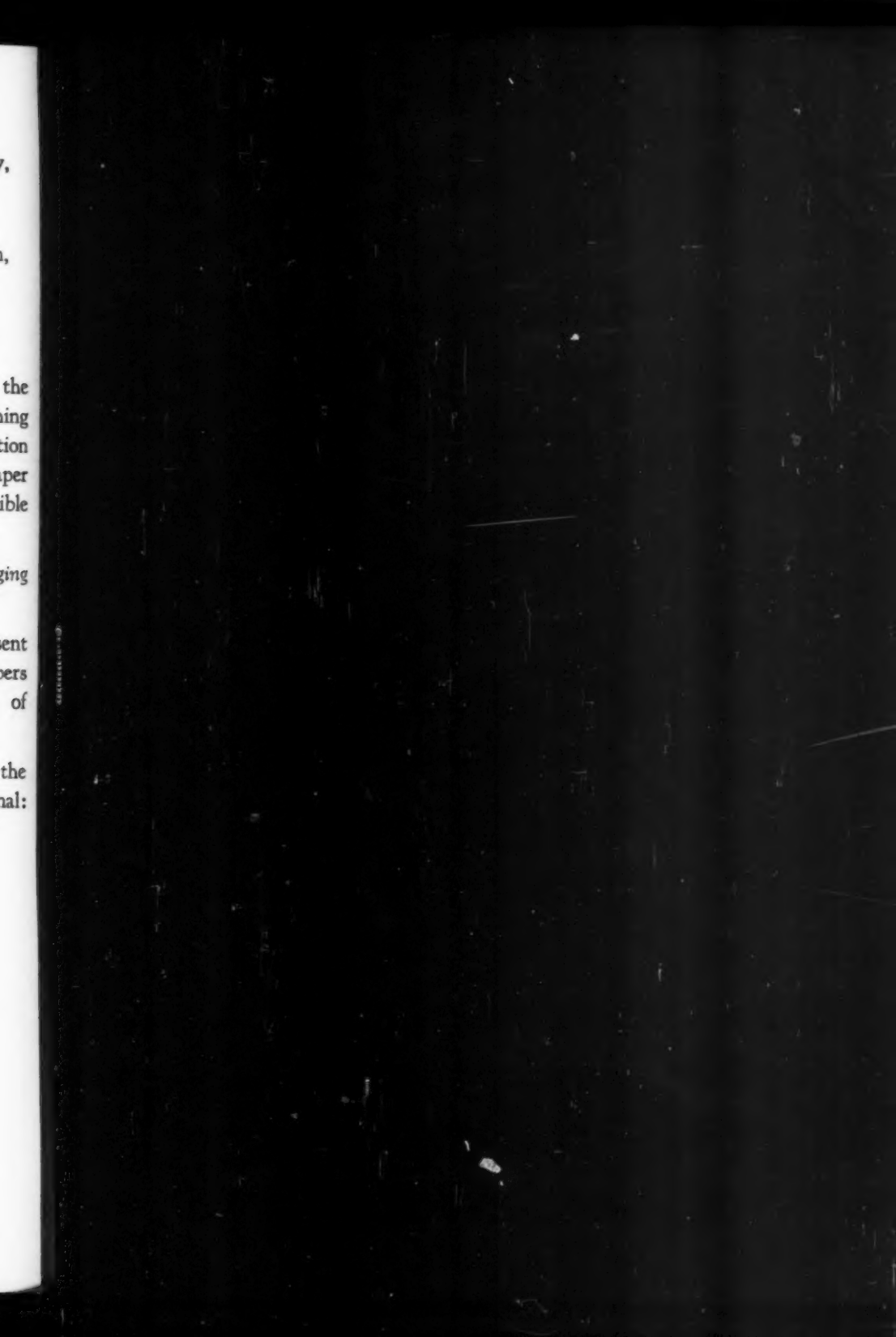

HYPERBOLIC MIXED PROBLEMS FOR HARMONIC TENSORS

G. F. D. DUFF

This paper may be regarded as a sequel to **(1)**, where the initial value or Cauchy problem for harmonic tensors on a normal hyperbolic Riemann space was treated. The mixed problems to be studied here involve boundary conditions on a timelike boundary surface in addition to the Cauchy data on a spacelike initial manifold. The components of a harmonic tensor satisfy a system of wave equations with similar principal part, and we assign two initial conditions and one boundary condition for each component. Our results will for the most part be deduced from the analogue of the mixed boundary value theorem of generalized potential theory, in which some components and some normal derivatives of components are specified on the boundary surface.

To establish this result in the hyperbolic case, we study linear systems of hyperbolic equations with similar principal part under corresponding boundary conditions of the two types mentioned. For this purpose we rely on **(7)** where the case of assigned boundary values is treated, and on the modifications **(5)** necessary for the normal derivative problem. We then apply the general existence theorem so found to a number of the special problems of harmonic tensors.

The analogy between mixed problems for hyperbolic equations and potential theory for elliptic equations has led some authors **(6; 9)** to study surface layer potentials in the hyperbolic case. However, this method has not yet been successfully used to find existence theorems for hyperbolic equations with variable coefficients. In this paper we shall encounter a situation familiar from potential theory—the occurrence of orthogonality conditions on the data in cases where the solution of the given problems is not unique. Again, however, indirect methods of constructing the solution when these conditions are satisfied seem to be necessary. This aspect of hyperbolic mixed problems arises only for systems of second order equations as the solutions of all such problems for a single equation are unique **(5)**.

The Maxwell equations can be conveniently written in the notation of harmonic tensors, and Theorem V below can be interpreted as an existence theorem for boundary problems concerning time-variable electromagnetic fields.

1. Mixed problems for systems of hyperbolic equations. Let a normal hyperbolic Riemannian space V_N be defined by the metric

$$ds^2 = a_{ik}\,dx^i\,dx^k, \tag{1.1}$$

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with Lorentz signature $(1, N-1)$ and containing one positive term. The associate tensor a^{ik} exists and is contravariant symmetric. We study systems of M normal hyperbolic linear partial differential equations in M dependent variables $u_r (r = 1, \ldots, M)$:

$$L_r(u_r) = a^{ik} \frac{\partial^2 u_r}{\partial x^i \partial x^k} + b^i \frac{\partial u_r}{\partial x^i} + \sum_{s=1}^{M} c_{rs} u_s = e_r. \tag{1.2}$$

Here the coefficients a^{ik} and b^i of the partial derivatives are the same in all equations of the system so that the interrelationship of the dependent variables is effected only by the coefficients c_{rs} $(r, s = 1, \ldots, M)$. Conditions of regularity for the coefficients will be discussed below.

The Cauchy initial value problem for (1.2) involves giving values for the u_r and their first normal derivatives on a surface $S: \phi(x^i) = 0$ which is spacelike:

$$a(\phi) = a^{ik} \frac{\partial \phi}{\partial x^i} \frac{\partial \phi}{\partial x^k} > 0.$$

This type of problem has been treated by many authors **(6, 8, 9, 10)** and can be regarded as solved. The solution is defined in the domain D_S of exclusive dependence on S which is bounded by a characteristic surface G passing through the rim C of the spacelike surface S. We note that the geometry of the characteristic surfaces of (1.2) is the same as for a single equation.

Let $T: \psi(x^i) = 0$ be a timelike surface $(a(\psi) < 0)$ which meets S in the rim C. We shall consider only that region D on the positive side of S in a chosen orientation of the timelike coordinate. The normal n to T has contravariant components $a^{ik}\, \partial\psi/\partial x^k$. We now divide the equations (1.2) into two sets. For $r = 1, \ldots, p$, we assign on T values of u_r:

$$u_r = f_r, \qquad r = 1, \ldots, p, \text{ on } T, \tag{1.3}$$

while for $r = p + 1, \ldots, N$ we assign values of the normal derivatives

$$\frac{\partial u_r}{\partial n} = g_r, \qquad r = p + 1, \ldots, M, \text{ on } T. \tag{1.4}$$

Together with Cauchy data on S, these conditions determine our mixed problem.

There are certain compatibility conditions to be satisfied by the datum functions f_r and g_r; these conditions ensure that u_r and its derivatives up to a certain order k are continuous across the characteristic surface G which is the locus of all discontinuities originating on the rim $C = S \cap T$. In practice it is convenient to subtract out the Cauchy data by first solving the Cauchy problem, and then subtracting the solutions from the u_r. The new problem so found has zero Cauchy data and the compatibility condition of order k is now the vanishing on C of the derivatives

$$\frac{\partial^k f_r}{\partial t^k}, \quad \frac{\partial^{k-1} g_r}{\partial t^{k-1}},$$

where t, a timelike variable, can be taken as equal to $\phi(x^i)$. Thus, the condition of order zero applies only to f_r. Consider first the case where the system (1.2), the data, and the surfaces S, T are analytic. The Cauchy problem can be solved and subtracted out. The domain R between G and T remains. To define the solutions in R we note that their values are now determined on G and these assigned values will be zero since the Cauchy data of the new problem are zero.

A new coordinate system in R is now introduced. Let G_t be a family of characteristic surfaces meeting T in hypercurves C_t, with $G_0 = G$, $C_0 = C$; and suppose that G_t is analytic. Since we can only secure a local existence theorem in the analytic case, it is enough to suppose that the G_t are defined in a neighbourhood of S and T. We now take G_t as coordinate hypersurface of a new variable $t = x'^N$, and also we set $x'^{N-1} = x' = \psi(x^i)$ so that the timelike surface T becomes a coordinate hyperplane. The reduction given by Hadamard **(6**, p. 76**)**, applied to the system (1.2), shows that in the new coordinate system the differential equations can be written

$$\frac{\partial^2 u_r}{\partial x \partial t} = L_1(u_r) + \sum_{s=0}^{M} c_{rs} u_s + e_r, \tag{1.5}$$

where we have dropped the primes on coordinates and coefficients. Here the new dependent variables u_r are equal to

$$u_r \exp[\int b^N \, dx^{N-1}]$$

as in Hadamard's reduction, and the operator $L_1(u_r)$ contains no differentiations with respect to t.

The auxiliary conditions are now

$$u_r = 0, \qquad r = 1, \ldots, M; \; t = 0, \tag{1.6}$$

and

$$u_r = f_r, \qquad r = 1, \ldots, p; \; x = 0, \tag{1.7}$$

with

$$\frac{\partial u_r}{\partial n} = b^N u_r + g_r, \qquad r = p + 1, \ldots, M; x = 0. \tag{1.8}$$

We shall suppose that the compatibility conditions of order up to $k > N + 2$ are satisfied, and it follows as in **(5)** that we can subtract from the u_r certain functions, analytic in R and C^k across G, which satisfy the above boundary conditions. This means that in (1.7) and (1.8) we can assume $f_r = 0$ and $g_r = 0$, while the non-homogeneous terms e_r in (1.5) absorb the extra terms appearing in the differential equations.

From **(5**, Lemma I**)** we see that the normal n to T is never tangent to the characteristic surfaces G_t. Consequently we may now write the homogeneous boundary conditions in the form

$$
\begin{aligned}
&u_r = 0; && r = 1, \ldots, p, \\
&\frac{\partial u_r}{\partial t} = \sum_{k=1}^{N-1} \beta^k \frac{\partial u_r}{\partial x^k} + h u_r, && r = p+1, \ldots, M.
\end{aligned} \tag{1.9}
$$

Here we have solved for the derivative with respect to t in the second group. These conditions apply for $x = 0$.

Let the unknowns u_r and all other functions in (1.5) and (1.9) be expanded in powers of t: e.g.

$$u_r = \sum_{n=0}^{\infty} u_{rn} t^n, \quad L_1(u_r) = \sum_{n=0}^{\infty} L_{1n}(u)\, t^n.$$

By a subscript n attached to a function we shall mean the coefficient of t^n in such an expansion. We now determine the u_{rn} recursively for $n = 1, 2, \ldots$. From (1.5) we have

$$n \frac{du_n}{dx} = L_{01}(u_{n-1}) + \ldots + \sum_s c_{rs0}\, u_{s,n-1} + \ldots + e_{n-1}, \tag{1.10}$$

where the terms omitted contain the $u_{r1}, \ldots, u_{rk}, \ldots$ up to the $u_{r\,n-2}$, and their derivatives with respect to $x^1, \ldots, x^{N-2}$, and $x = x^{N-1}$. The boundary conditions (1.9) now read

$$
\begin{aligned}
&u_{rn} = 0, && r = 1, \ldots, p; \\
&n\, u_{rn} = \sum_{k=1}^{N-1} \beta^k{}_0 \frac{\partial u_{r,n-1}}{\partial x^k} + h_0\, u_{r,n-1} + \ldots, && r = p+1, \ldots, M.
\end{aligned} \tag{1.11}
$$

To satisfy (1.6) we take $u_{r0} = 0$. Thus the u_{rn} are determined by integration with respect to x. The expressions so found contain additions, multiplications and integrations, all of which operations preserve the relation of dominance between power series in the variables $x^1, \ldots, x^{N-1}, t$. Thus by dominating the coefficients in (1.5) and (1.9) we would find a series solution dominating the original one.

By choosing a single function U to dominate all of the u_r, we can reduce the proof that the dominating series converges to the case considered in **(5)**. This can be done by choosing a single dominating coefficient for each of the combinations

$$\sum_s c_{rs} \qquad (s = 1, \ldots, M)$$

which would appear in (1.5) if all u_r were equal, and a common dominant for the e_r. The dominating series in **(5)** has a positive radius of convergence in each variable, and has positive coefficients, so that (1.6) and the first of (1.9) are dominated. This series also satisfies for all $x^1, \ldots, x^{N-1}, t$ a relation **(5, 2.27)** which is equivalent to the domination of the second group of conditions (1.9). We may therefore conclude that a local analytic solution of (1.5), (1.6), (1.7), (1.8) exists, with a radius of convergence independent of the datum functions. Referring back to the original problem, we see that the solution here is analytic except possibly on G where it is of class C^k.

To extend the domain of the solution, or to lighten the hypothesis of analyticity, certain estimates of the square integrals of the solutions and their derivatives are needed. Except for one point the procedure follows that in **(5; 7)**, so we shall not give details. The procedure involves integrations by parts carried out over (1.2) after multiplication by a first derivative of u in a certain timelike direction. A different choice of this direction is necessary for the two groups $r = 1, \ldots, p$ and $r = p + 1, \ldots, M$. For $r = 1, \ldots, p$ with the Dirichlet-type condition as in **(7)**, this direction must cross T inwards with increasing time, while for $r = p + 1, \ldots, M$ and the normal derivative condition as in **(5)**, the direction must cross T outwards with increasing time. The result of these calculations is estimates of the k-norms $||u_r||_k$ as defined in **(7)** in terms of the functions e_r, the Cauchy data, and the given data in (1.3) and (1.4). Applying the lemmas of **(7)** to secure convergence we are able to extend the domain of definition of the solution in the direction of the future, and to show the existence of solutions in the non-analytic case.

Finally, we shall remove the compatibility conditions on the rim C, with the exception of the first of these conditions for the group $r = 1, \ldots, p$. This is done, as in **(5, §6)**, by writing $u_r = u_{1r} + v_r$, and requiring that $L_r(v_r)$ be of class C^k in the entire domain while v_r satisfies boundary conditions such that the conditions for u_{1r} are compatible of order k. We assume that the differential equation and data are of class C^{2k}, and construct v_r as the first k terms of the series expansion in the analytic case. The problem for u_{1r} is now seen to be C^k and so is reduced to the preceding case.

We state the general result as an existence theorem; the following sections contain the applications to harmonic tensors.

EXISTENCE THEOREM. *Let the coefficients in* (1.2) *be of class* C^{2k} $(k \geqslant N + 2)$ *and let the Cauchy data together with the data in* (1.3) *and* (1.4) *be of class* C^{2k+N}. *Let the data in* (1.3) *and* (1.4) *satisfy the compatibility conditions up to order* q *inclusive,* $q \leqslant k - N$. *Then there exists a unique solution* u_r *of* (1.2), (1.3) *and* (1.4), *together with the given Cauchy data, and this solution is of class* C^{k-N} *except across* G *where it is of class* C^q.

The hypotheses stated in the above theorem have been simplified and will be applied with $q = 1$ or 2 to the theory of harmonic tensors. However less restrictive hypotheses, and a slightly stronger conclusion, are possible as the situation is effectively the same as in **(5, §6)**.

We shall use the uniqueness of the solution several times below, and so we state here the conditions of regularity which a solution u_r, satisfying the homogeneous equations and auxiliary conditions, must satisfy for this purpose. Let u_r be of class C^2, except possibly across G, where continuity only is assumed. Since the homogeneous problem is compatible of every order k, it follows that the first and second derivatives of u_r across G are also continuous **(5, §1)**. Thus u_r is C^2 and the estimates referred to above show that $u_r = 0$.

The statements of the theorems below will specify the value of q in each case. For brevity, we now assume once for all that $k > N + 2$ and all solutions are of class C^{k-N} except possibly across G.

2. Application to harmonic tensors. We recall here the main points of the necessary notation **(1)**. A differential form

$$\phi = \phi_{i_1,\ldots,i_p}\, dx^{i_1} \wedge \ldots \wedge dx^{i_p}, \tag{2.1}$$

where summation with $i_1 < \ldots < i_p$ is understood, is based on a totally skew symmetric tensor $\phi_{i_1,\ldots,i_p}$ of rank p. The differentials dx^i are combined by exterior multiplication which has the anti-commuting property $dx^i \wedge dx^j = -dx^j \wedge dx^i$. The differential of ϕ is a form of degree $p + 1$:

$$d\phi = (d\phi_{i_1,\ldots,i_p}) \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p}. \tag{2.2}$$

and

$$(d\phi)_{i_1\ldots i_{p+1}} = \Gamma_{i_1\ldots i_{p+1}}^{j(j_1\ldots j_p)} \frac{\partial}{\partial x^j} \phi_{(j_1\ldots j_p)}, \tag{2.3}$$

where Γ is the skew symmetrized Kronecker delta of rank $p + 1$, and the brackets indicate $j_1 < \ldots < j_p$. The dual $*\phi$ is an $(N - p)$-form with components defined by contraction with the permutation symbol of order N. Then the coderivative $\delta\phi$, of degree $p - 1$, is defined by

$$\delta\phi = (-1)^{Np+p+1} * d * \phi.$$

The identities

$$d.d\phi \equiv 0, \qquad **\phi \equiv (-1)^{Np+p}\,\phi, \qquad \delta.\delta\phi \equiv 0, \tag{2.4}$$

hold.

If $d\phi = 0$, ϕ is said to be closed, and if $\phi = d\chi$, derived; and dually $\delta\phi = 0$, $\phi = \delta\chi$ describe coclosed and coderived forms.

The Laplacian operator Δ is given by

$$\Delta = \delta d + d\delta, \tag{2.5}$$

and if the equation $\Delta\phi = 0$ of harmonic tensors is written out in component form it is of the type of (1.2), with $M = \binom{N}{p}$ equations. The coefficients c_{rs} then consist of contractions of the Riemann curvature tensor of the metric (1.1).

On any hypersurface S with equation, say $x^m = 0$, a p-form ϕ induces a p-form $t\phi$, which contains those components free of the index m, and is known as the tangential part of ϕ. The residual or normal part, is denoted by $n\phi$, and has the factor dx^m. Thus we shall write

$$\phi = t\phi + n\phi = t\phi + \tilde{n}\phi \wedge dx^m, \tag{2.6}$$

which decomposition is invariant only on S. The dual interchanges t and n; thus $*t = n*$, $*n = t*$. If $t\phi$ and $n\phi$ are assigned on S, then $td\phi = d_S t\phi$ and $n\delta\phi$ are determined, but $nd\phi$ and $t\delta\phi$ can be assigned independently.

From Stokes formula

$$\int_C d\phi = \int_{bC} \phi, \tag{2.7}$$

which is valid for any $(p+1)$-chain C with boundary bC, we observe that the period integral $\int_z \phi$ of a closed form ϕ depends only on the homology class of Z. A closed form with vanishing absolute periods on a manifold with boundary is derived **(2**, Theorem 4**)**.

To distinguish among the various cases which follow, we shall give to each problem the name of the problem in elliptic generalized potential theory with the corresponding boundary conditions. The Cauchy data $t\phi$, $n\phi$, $t\delta\phi$, $nd\phi$ on an initial surface S will be assigned in each case. We may allow the initial surface S to be a multiply-connected manifold, bounded by a manifold $C = S \cap T$. We shall, however, assume that any non-bounding cycle of the region enclosed by S and T is homologous to a cycle of S, and that any cycle of T is homologous to a cycle of C; and vice versa. In effect, we take the domain to be topologically uniform in the time dimension. The data will always be assumed to satisfy the conditions of the preceding existence theorem.

The Dirichlet problem involves values of $t\phi$ and $n\phi$ on T; thus we shall invoke the existence theorem with $p = M$ so that all equations belong to the first group. Then we have

THEOREM I. *There exists a unique harmonic form ϕ with given Cauchy data on S and given values of $t\phi$ and $n\phi$ on T.*

Here the compatibility condition of the general existence theorem implies that $t\phi$ and $n\phi$ given on T should have values on C determined by the Cauchy data $t\phi$ and $n\phi$ on C.

In the mixed problem of generalized potential theory the boundary data are values of $t\phi$ and $t\delta\phi$, or alternatively and dually **(3)**, of $n\phi$ and $nd\phi$. From (2.3) we deduce the formula

$$\bar{n}d\phi = (-1)^{p+1} \frac{\partial t\phi}{\partial x^m} + d_s \bar{n}\phi, \tag{2.8}$$

the component form of which is found by setting $i_{p+1} = m$ and separating the term $j = m$ from the others. Thus, if $n\phi$ and $nd\phi$ are given, the first normal derivative of $t\phi$ is determined and so the data in this case amount to values of $n\phi$ and normal derivatives of components of $t\phi$. We therefore choose the two groups of the system (1.2) accordingly, and conclude with the

THEOREM II. *There exists a unique harmonic form ϕ with given Cauchy data on S and given values of $n\phi$ and $nd\phi$ (or $t\phi$ and $t\delta\phi$) on T.*

Here the compatibility condition on C requires that $n\phi$ (or $t\phi$, as the case may be) should have values on C in accordance with the Cauchy data. We shall later need solutions of class C^1 or C^2 and so will assume the compatibility

conditions of order 2 and 3 as necessary. These restrict both $n\phi$ and $nd\phi$ (or $t\phi$ and $t\delta\phi$) in the neighbourhood of the rim C.

The uniqueness of this solution in the hyperbolic problem is in sharp contrast to the elliptic case where there exist eigenforms in number determined by the p-dimensional connectivity (Betti number) of the domain.

In both of these theorems we could state results for the non-homogeneous differential equations

$$\Delta\phi = \rho, \tag{2.9}$$

by solving a Cauchy problem for (2.9) in a larger domain and subtracting out the solution.

3. Particular cases of the mixed problem. The result of Theorem II admits of a number of special cases in which the governing differential equation $\Delta\phi = \rho$ is replaced by a pair of equations. The boundary conditions also take various forms; these have been studied in the elliptic case **(4)**. We begin with co-closed harmonic forms which satisfy $\delta\phi = 0$, $\delta d\phi = 0$. The corresponding non-homogeneous equations will be treated.

For our first theorem relative to coclosed harmonic forms, we assign initial data, indicated by the subscript 0, and values for the tangential part of ϕ on T. The differential equations are

$$\delta d\phi = \rho, \quad \delta\phi = \sigma, \tag{3.1}$$

where $\rho = \rho_p$ and $\sigma = \sigma_{p-1}$, are assigned coderived forms. The initial values to be given are such that together with (3.1) Cauchy data for the solution can be obtained. Thus we prescribe $t\phi_0$, $t*\phi_0$ and $t*d\phi_0$, indicating initial values with the subscript 0. In order that these data should be compatible with (3.1) we must require

$$\begin{aligned} d_S t*\phi_0 &= (-1)^N t*\sigma_0, \\ d_S t*d\phi_0 &= (-1)^N t*\rho_0. \end{aligned} \tag{3.2}$$

We then set $t\delta\phi_0 = t\sigma_0$ and so obtain complete Cauchy data.

In the boundary condition

$$t\phi = \theta, \qquad \text{on } T, \tag{3.3}$$

where θ is a given p-form, the first two compatibility conditions must be satisfied. That is, θ and its first timelike derivative on C shall have values given by the Cauchy values there.

THEOREM III. *Let $t\phi_0$, $t*\phi_0$ and $t*d\phi_0$ be assigned on S subject to* (3.2). *Let the boundary condition* (3.3) *satisfy the first two compatibility conditions on C. Then there exists a unique solution of* (3.1) *satisfying these conditions, and this solution is of class C^1.*

To prove this we set up the mixed problem

$$\Delta\phi = \rho + d\sigma, \tag{3.4}$$

with boundary conditions

$$t\phi = \theta, \; t\delta\phi = t\sigma, \qquad \text{on } T, \tag{3.5}$$

and the Cauchy data introduced above. The values of $t\phi$ are compatible for $k = 0$ and 1, while the values of $t\delta\phi$ are compatible for $k = 1$ since $\delta\phi_0 = \sigma_0$ in view of the first of (3.2) and the definition $t\delta\phi_0 = t\sigma_0$. The general existence theorem now asserts that ϕ exists and is of class C^1.

We now define

$$\psi = \delta\phi - \sigma \tag{3.6}$$

and note that ψ is continuous across G and is elsewhere of class C^{k-N-1}. Also

$$d\psi = d\delta\phi - d\sigma = \rho - \delta d\phi$$

is coderived and so $\delta d\psi = 0$. Since $\delta\psi = \delta.\delta\phi - \delta\sigma = 0$ it follows that $\Delta\psi = 0$, We now show that ψ has zero Cauchy data. From $t\delta\phi_0 = t\sigma_0$, we get $t\psi_0 = 0$. and likewise

$$t*\psi_0 = t*\delta\phi_0 - t*\sigma_0 = (-1)^N d_S t*\phi_0 - t*\sigma_0 = 0,$$

from the first of (3.2). Also

$$\begin{aligned} t*d\psi_0 = t*(\rho - \delta d\phi) &= t*\rho_0 - (-1)^N td*d\phi_0 \\ &= t*\rho_0 - (-1)^N d_S t*d\phi_0 = 0, \end{aligned}$$

from the second of (3.2). Finally $t\delta\psi_0 = 0$ since ψ is coclosed, and thus the Cauchy data for ψ are zero. Turning to the boundary conditions, we find $t\psi = 0$ on T from the second of (3.5) while $t\delta\psi \equiv 0$. Thus ψ satisfies homogeneous boundary conditions of the mixed type. From the uniqueness of the solution in Theorem II we conclude that ψ vanishes identically. Together with (3.4) and (3.6) this establishes (3.1) and concludes the proof of the theorem.

Another set of boundary conditions applicable to the differential equations (3.1) are found from the dual mixed problem: thus

$$t*\phi = \xi, \quad t*d\phi = \eta, \qquad \text{on } T \tag{3.7}$$

are specified. The restrictions

$$d_T\xi = d_T t*\phi = (-1)^N t*\delta\phi = (-1)^N t*\sigma \tag{3.8}$$

and

$$d_T\eta = d_T t*d\phi = (-1)^N t*\delta d\phi = (-1)^N t*\rho \tag{3.9}$$

are necessary in view of (3.1). The initial values are as in the preceding theorem. Again, we require that the compatibility conditions of orders $k = 0$ and 1 for ξ and order $k = 1$ for η should hold. Thus the value and first derivative of ξ and the value of η on C must agree with those calculated from the Cauchy data.

THEOREM IV. *Let $t\phi_0$, $t*\phi_0$ and $t*d\phi_0$ be assigned on S subject to* (3.2). *Let the data in* (3.7) *satisfy* (3.8) *and* (3.9). *Then there exists a unique solution of* (3.1) *satisfying these conditions, and this solution is of class C^1.*

For the proof we again solve (3.4), this time with boundary conditions (3.7) and the above Cauchy data. The compatibility conditions of orders $k = 0$ and 1 are satisfied and a C^1 solution ϕ exists, by the dual of Theorem II. Again, let $\psi = \delta\phi - \sigma$, which is continuous, and let us show that ψ vanishes. As before, $\Delta\psi = 0$ and ψ has zero Cauchy data. On T we have

$$\begin{aligned} t*\psi &= t*(\delta\phi - \sigma) = (-1)^N d_T t*\phi - t*\sigma \\ &= (-1)^N d_T\xi - t*\sigma = 0 \end{aligned}$$

from (3.8). Also

$$\begin{aligned} t*d\psi &= t*(d\delta\phi - d\sigma) = t*(\rho - \delta d\phi) \\ &= t*\rho - (-1)^N d_T t*d\phi \\ &= t*\rho - (-1)^N d_T\eta = 0 \end{aligned}$$

from (3.9). Thus ψ satisfies the homogeneous dual mixed boundary conditions and so must vanish everywhere. This proves the theorem.

We turn now to the differential equations associated with harmonic fields—forms which are both closed and coclosed. The corresponding nonhomogeneous equations are

$$d\phi = \rho, \quad \delta\phi = \sigma, \tag{3.10}$$

where $\rho = \rho_{p+1}$ must be derived and $\sigma = \sigma_{p-1}$ coderived.

For initial values we now assign $t\phi_0$ and $t*\phi_0$ subject to the necessary conditions

$$d_s t\phi_0 = td\phi_0 = t\rho_0 \tag{3.11}$$

and

$$d_s t*\phi_0 = td*\phi_0 = (-1)^N t*\delta\phi_0 = (-1)^N t*\sigma_0. \tag{3.12}$$

We can now obtain Cauchy data for this problem by adjoining to these values

$$t*d\phi_0 = t*\rho_0 \text{ and } t\delta\phi_0 = t\sigma_0.$$

The appropriate boundary condition is

$$t\phi = \xi, \tag{3.13}$$

where the necessary condition

$$d_T\xi = td\phi = t\rho \tag{3.14}$$

holds. This restriction on ξ will be examined in more detail below; we remark that ξ shall also satisfy two compatibility conditions in relation to the above Cauchy data.

THEOREM V. *Let $t\phi_0$ and $t*\phi_0$ be assigned on S subject to* (3.11) *and* (3.12). *Let ξ in* (3.13) *satisfy* (3.14) *and the first two compatibility conditions. Then*

there exists a unique solution of (3.10) *which satisfies these conditions and this solution is of class* C^1.

To prove this we establish the mixed problem

$$\Delta\phi = \delta\rho + d\sigma \tag{3.15}$$

with the Cauchy data given above, and with boundary conditions

$$t\phi = \xi, \quad t\delta\phi = t\sigma. \tag{3.16}$$

The data being compatible of orders $k = 0$ and 1, the solution ϕ is C^1. Let

$$\psi = \delta\phi - \sigma, \quad \chi = d\phi - \rho; \tag{3.17}$$

both of these forms are continuous across G.

We will show that both ψ and χ vanish identically. Taking first ψ, we have $\delta\psi = \delta.\delta\phi - \delta\sigma = 0$ and

$$d\psi = d\delta\phi - d\sigma = \delta\rho - \delta d\phi,$$

so that $\delta d\psi = 0$ and $\Delta\psi = 0$. The Cauchy data of ψ are now to be found. First, $t\psi_0 = t\delta\phi_0 - t\sigma_0 = 0$ from the construction of $t\delta\phi_0$. Next

$$t*\psi_0 = t*\delta\phi_0 - t*\sigma_0 = (-1)^N d_S\, t*\phi_0 - t*\sigma_0 = 0,$$

from (3.12). Then $t\delta\psi_0 = 0$ since $\delta\psi \equiv 0$, and finally

$$t*d\psi_0 = t*(\delta\rho_0 - \delta d\phi_0) = (-1)^N d_S\, t*(\rho_0 - d\phi_0) = 0,$$

according to the construction of $t*d\phi_0$. Turning to the boundary conditions satisfied by ψ, we have $t\psi = t\delta\phi - t\sigma = 0$ from the second of (3.16), while $t\delta\phi = 0$. Thus ψ satisfies homogeneous mixed boundary conditions as well and so vanishes as required. This proves the second of (3.10).

To show that χ is zero, we have $d\chi = d.\, d\phi - d\rho = 0$, while

$$\delta\chi = \delta d\phi - \delta\rho = d\sigma - d\delta\phi = -d\psi = 0$$

so $\Delta\chi = 0$. The Cauchy data for χ are:

$$t\chi_0 = t\delta\phi - t\rho = d_S t\phi - t\rho = 0$$

by (3.11);

$$t*\chi_0 = t*d\phi_0 - t*\rho_0 = 0$$

by the way $t*d\phi_0$ was defined;

$$t\delta\chi_0 = 0, \quad t*d\chi_0 = 0.$$

The boundary conditions for χ are

$$t\chi = td\phi - t\rho = d_T t\phi - t\rho = d_T\xi - t\rho = 0$$

by (3.14); and $t\delta\chi = 0$ since χ is coclosed as demonstrated above. Since χ now satisfies all conditions of the homogeneous mixed problem it vanishes

identically and this proves the first of (3.10). Thus ϕ satisfies both differential equations and the proof of the theorem is complete.

The restriction (3.14) on the datum form ξ will now be examined in more detail in order to find just what components can be freely assigned on T. For simplicity we shall suppose the differential equations homogeneous and the initial values zero. We assert that in these circumstances (3.14) together with the first compatibility condition imply that ξ is a derived form on T. Indeed if $d_T\xi = 0$ the periods of ξ on p-cycles of T are well defined, and by our general topological hypothesis of §2, any cycle of T is homologous in T to a cycle of C. On this cycle the period is zero since ξ vanishes there. By **(2**, Theorem 4**)**, it follows that ξ is derived on T, say $\xi_p = d_T\zeta_{p-1}$.

Let C_t be a family of spacelike $(N-2)$-dimensional surfaces covering T, with $C_0 = C$, and let ξ be decomposed into tangential and normal parts relative to C_t. We shall, for convenience, refer to these as the spacelike and timelike parts of ξ, respectively. We now show that due to (3.14) the spacelike part is determined by the timelike part. Indeed

$$(d_T\xi)_{ti_1\ldots i_p} = \frac{\partial \xi_{i_1\ldots i}}{\partial t} - \ldots + \ldots = 0$$

where the subscript t refers to the timelike parameter of the family C_t; and where the terms not shown are derivatives of components of the timelike part of ξ. Since ξ is given for $t = 0$ by the compatibility condition, we may determine

$$\xi_{i_1\ldots i_p}, \qquad i_1, \ldots, i_p \neq t$$

by integration over t. That is, the spacelike part, having $\binom{N-2}{p}$ components, is determined by the timelike part, which has $\binom{N-2}{p-1}$ components. That the components of the timelike part can be freely assigned is evident from the component form of $\xi_p = d_T\zeta_{p-1}$, namely

$$\xi_{ti_1\ldots i_{p-1}} = \frac{\partial \xi_{i_1\ldots i_{p-1}}}{\partial t} - \ldots + \ldots,$$

since the time-derivatives can be chosen at will.

Maxwell's equations in empty space have the form

$$dF_2 = 0, \quad \delta F_2 = J_1,$$

where, in locally Cartesian coordinates with Lorentz metric,

$$F_2 = H_1 dy \wedge dz + H_2 dz \wedge dx + H_3 dx \wedge dy + (E_1 dx + E_2 dy + E_3 dz) \wedge dt,$$

and

$$J_1 = J_x dx + J_y dy + J_z dz + \rho dt.$$

Theorem V applies to determine F_2 if J_1 is known and the auxiliary conditions are specified as follows. For $t = 0$, assign tF_2 and $t{*}F_2$, that is, values of the components of $\underline{E}$ and $\underline{H}$. These are subject to (3.11) and (3.12) which become

$$\operatorname{div} \underline{H} = 0, \quad \operatorname{div} \underline{E} = \rho$$

(ρ denoting charge density), respectively. Thus two components of each vector can be freely specified. For the boundary conditions on a surface with local equation $x = 0$, we have

$$tF_2 = H_1 dy \wedge dz + E_2 dy \wedge dt + E_3 dz \wedge dt,$$

the first term on the right being the spacelike part. Thus the two tangential components of $\underline{E}$ are to be assigned, and determine the field uniquely. Applying the theorem to the dual of F_2, we should take the tangential components of $\underline{H}$.

As another direct application of Theorem II, we consider the equations

$$\delta d\phi = \rho, \quad d\delta\phi = \sigma, \tag{3.18}$$

associated with the class of harmonic tensors called biharmonic fields **(4)**. Here $\rho = \rho_p$ is coderived and $\sigma = \sigma_p$ derived. For initial values we now assign the four Cauchy data $t\phi_0$, $t*\phi_0$, $t*d\phi_0$ and $t\delta\phi_0$. However the latter two are subject to the conditions

$$d_S t\delta\phi_0 = t\sigma_0; \quad d_S t*d\phi_0 = (-1)^N t*\rho_0. \tag{3.19}$$

The boundary values

$$t\phi = \xi, \quad t\delta\phi = \eta, \qquad \text{on } T, \tag{3.20}$$

shall be required, where η satisfies

$$d_T \eta = t\sigma, \qquad \text{on } T. \tag{3.21}$$

In the proof we shall need a uniqueness theorem for second differentials of ϕ and so we assume compatibility conditions for ξ and η of orders $k = 0, 1$ and 2 on the rim C.

THEOREM VI. *Let Cauchy data on S be assigned subject to* (3.19). *Let ξ and η satisfy* (3.21) *and the first three compatibility conditions on C. Then there exists a unique solution of* (3.18) *satisfying the initial and boundary conditions, and this solution is of class C^2.*

The theorem is proved by constructing ϕ as a solution of

$$\Delta\phi = \rho + \sigma, \tag{3.22}$$

with mixed boundary conditions (3.20). In view of the compatibility conditions assumed for ξ and η, the solution is of class C^2 across G. Now let

$$\psi = \delta d\phi - \rho = \sigma - d\delta\phi;$$

this form is continuous, and being both derived and coderived is also harmonic. Now the Cauchy data for ψ vanish since

$$t\psi_0 = t\sigma_0 - d_T t\delta\phi_0 = 0,$$
$$t*\psi_0 = t*\delta d\phi_0 - t*\rho_0 = (-1)^N d_S t*d\phi_0 - t*\rho_0,$$

by (3.19); while $t*d\psi_0$ and $t\delta\psi_0$ are clearly zero. On the boundary T, we have

$$t\psi = t\sigma - d_T t\delta\phi = t\sigma - d_T\eta = 0,$$

from (3.21), while $t\delta\psi$ is zero identically. Thus ψ vanishes and (3.18) hold, which concludes the proof.

4. Orthogonality conditions. The problems studied in this section do not have unique solutions and moreover the data must satisfy certain integral conditions of the orthogonal type familiar from potential theory or integral equations. As above, we consider a spacelike surface S bounded by its intersection C with a timelike surface T. We denote by Σ any spacelike surface lying in the region of definition of the solutions, such that S, T, and Σ bound a portion of the region. More precisely, we denote by S_Σ and T_Σ the parts of these surfaces which, together with Σ, have this property.

The Neumann problem of generalized potential theory concerns solutions of $\Delta\phi = 0$ with given values of $t*d\phi$ and $t\delta\phi$ on the boundary. We assume these conditions to hold on T and take zero Cauchy data for simplicity. Now any harmonic field ($d\tau = 0$, $\delta\tau = 0$) with zero Cauchy data is an eigenform of this problem and from Theorem V we see that an infinity of linearly independent eigenforms τ exist. In the elliptic case these furnish the orthogonality conditions directly. If we write down Green's theorem for the domain bounded by S_Σ, T_Σ and Σ in the form

$$(4.1)\quad (d\tau, d\phi) + (\delta\tau, \delta\phi) - (\tau, \Delta\phi) = \int_{S_\Sigma + T_\Sigma + \Sigma} (\tau \wedge *d\phi - \delta\phi \wedge *\tau),$$

where (α, β) denotes the volume integral of $\alpha \wedge *\beta$, **(1)**, we see that the terms on the left vanish, and that the integral over S is zero since ϕ has zero Cauchy data. Thus, if τ is a harmonic field, with zero tangential and normal part on Σ, defined in the region bounded by S_Σ, T_Σ and Σ, the integral over Σ disappears and we have

$$(4.2)\qquad \int_{T_\Sigma} (\tau \wedge *d\phi - \delta\phi \wedge *\tau) = 0.$$

These are the orthogonality conditions of the Neumann problem and they arise from the eigenforms of the backward problem for harmonic fields with "final" surface Σ. We remark that in (4.2) $t*\tau$ and $t\tau$ are not independent since, by Theorem V, values of $t\tau$ alone determine τ as a harmonic field.

The problem to be solved is then to find a harmonic form ϕ with zero Cauchy data on S and

$$(4.2)'\qquad t*d\phi = \xi, \quad t\delta\phi = \eta \qquad \text{on } T,$$

where

$$(4.3)\qquad \int_{T_\Sigma} (\tau \wedge \xi - \eta \wedge *\tau) = 0$$

for every backward harmonic field τ vanishing on Σ, for every such surface Σ. Let ξ and η vanish on C.

We begin by solving the mixed problem with $t\delta\phi = \eta$; thus let $\Delta\phi_1 = 0$, let ϕ_1 have zero Cauchy data, let $t\phi_1 = 0$ on T and $t\delta\phi_1 = \eta$. By Theorem II there exists a C^1 form ϕ_1 satisfying these conditions. Now also

$$\int_{T_\Sigma}(\tau \wedge *d\phi_1 - \delta\phi_1 \wedge *\tau) = \int_{T_\Sigma}(\tau \wedge *d\phi_1 - \eta \wedge *\tau) = 0,$$

and if we subtract these from (4.3) we get

$$\int_{T_\Sigma}\tau \wedge (\xi - t*d\phi_1) = 0. \tag{4.4}$$

The following lemma is now necessary.

LEMMA. *Let ζ be a form of degree $N - p + 1$ defined on T, and zero on $C = S \cap T$. Let*

$$\int_{T_\Sigma}\tau \wedge \zeta = 0 \tag{4.5}$$

for all closed forms τ defined on T, and vanishing on $C_\Sigma = T \cap \Sigma$. Then ζ is closed on T: $d_T\zeta = 0$.

To prove this we let ψ be a solution of

$$\delta d\psi = 0, \quad \delta\psi = 0,$$

having zero Cauchy data on Σ, and defined in our region. According to Theorem III, we may choose $t\psi$ on T arbitrarily (of class C^k) except that $t\psi$ and its first timelike derivative must vanish on C_Σ. Then $\tau = d\psi$ is a harmonic field, with zero Cauchy data on Σ and so vanishing on C_Σ. Also $t\tau$ is closed on T so we may insert $t\tau = td\psi$ in (4.5). Since $S_\Sigma + T_\Sigma + \Sigma$ bounds a region, we have by Stokes formula

$$0 = \int_{S_\Sigma+T_\Sigma+\Sigma} d(t\psi \wedge \zeta) = \int_{S_\Sigma+T_\Sigma+\Sigma} t\tau \wedge \zeta \pm \int_{S_\Sigma+T_\Sigma+\Sigma} t\psi \wedge d_T\zeta.$$

Here we have extended the definition of ζ to a form vanishing on S and defined throughout the region. By (4.5) and in view of our hypotheses, the first integral on the right vanishes. The second reduces to an integral over T_Σ and so we have

$$\int_{T_\Sigma} t\psi \wedge d_T\zeta = 0.$$

But $t\psi$ is arbitrary except for its values on C_Σ; we conclude that $d_T\zeta = 0$ as required. From (4.4) we now infer

$$d_T(\xi - t*d\phi_1) = 0, \tag{4.6}$$

since we can construct harmonic fields τ satisfying the condition of the lemma, and since $\xi - t*d\phi_1$ vanishes on C. By Theorem IV we construct a second form χ, with zero Cauchy data on S, such that

(4.7) $$\delta d\chi = 0, \quad \delta\chi = 0,$$

and

(4.8) $$t*\chi = 0, \quad t*d\chi = \xi - t*d\phi_1, \qquad \text{on } T.$$

Here the necessary conditions (3.8) and (3.9) hold, the latter in view of (4.6). The data in (4.8) are also compatible of the order stated in Theorem IV since $\xi - t*d\phi_1$ vanishes on S. Therefore such a C^1 form χ exists.

Now let

(4.9) $$\phi = \phi_1 + \chi;$$

we have $\Delta\phi = 0$ while ϕ has zero Cauchy data on S. Also

(4.10) $$t\delta\phi = t\delta\phi_1 + t\delta\chi = \eta + 0 = \eta,$$

while

(4.11) $$\begin{aligned} t*d\phi &= t*d\phi_1 + t*d\chi \\ &= t*d\phi_1 + \xi - t*d\phi_1 = \xi. \end{aligned}$$

This proves

THEOREM VII. *There exists a harmonic form ϕ with zero Cauchy data satisfying* (4.2) *subject to the orthogonality conditions* (4.3); *and this form is C^1.*

The above proof also shows that the orthogonality conditions are sufficient to determine $d_T\xi$ when η is given. Therefore the global conditions (4.3) are equivalent to this pointwise condition, which however cannot be stated explicitly without introducing as we have done a solution ϕ_1 of the mixed problem.

When the data ξ and η satisfy q compatibility conditions, which we shall not need to write out explicitly, then the solution constructed above is of class C^q. We shall take $q > 2$ in the following second theorem for the biharmonic field equations. The differential equations shall be (3.18) and the Cauchy data are again subject to (3.19). The boundary conditions are (4.2) where ξ and η each satisfy two compatibility conditions with respect to the given equations and initial values. In addition the conditions

(4.12) $$d_T\xi = (-1)^N t*\rho, \quad d_T\eta = t\sigma$$

are clearly necessary.

THEOREM VIII. *Let Cauchy data on S be assigned subject to* (3.19). *Let ξ and η satisfy* (4.12) *and compatibility conditions of orders $k = 0$ and 1 on C. Then there exists a solution of* (3.18) *satisfying the initial and boundary conditions, and this solution is of class C^2.*

To prove this we again consider the equation $\Delta\phi = \rho + \sigma$ with the given Cauchy data and the boundary conditions (4.2). For this nonhomogeneous equation the orthogonality conditions take the form

(4.13) $$(\rho + \sigma, \tau) + \int_{T_2+S_2+\Sigma} (\tau \wedge *d\phi - \delta\phi \wedge *\tau) = 0.$$

As in (3.18) we must have ρ coderived and σ derived; from (3.19) we see that we can write $\rho = \delta\theta$ and $\sigma = d\zeta$ where

$$t\zeta = t\delta\phi_0, \quad (-1)^N t*\theta = t*d\phi_0. \tag{4.14}$$

Making use of Stokes formula we transform the volume integral and obtain for the left hand side in (4.13)

$$\int_{T_\Sigma + S_\Sigma} (\tau \wedge [t*d\phi - t*\theta] - [t\hat{\iota}\phi - t\zeta] \wedge *\tau) = 0. \tag{4.15}$$

According to (4.14) the integrand is zero over S_Σ, and from (4.12) it follows (as a converse of the lemma) that the integrals over T_Σ vanish. Thus the orthogonality conditions are satisfied and a solution, which will be of class C^2, exists. We can now set

$$\psi = \delta.d\phi - \rho = \sigma - d\delta\phi$$

and show that ψ is harmonic with zero Cauchy data. From the second of (4.12) we find $t\psi = 0$ on T while $\delta\psi \equiv 0$ implies $t\delta\psi = 0$ on T. Thus ψ vanishes identically and this proves the stated result. Though these differential equations are very little stronger than the single Laplace or Poisson equation, the pointwise conditions on the data can be stated directly and no orthogonality condition intervenes. The eigenforms of this problem are again the harmonic fields with vanishing Cauchy data.

As a final example we discuss another problem wherein orthogonality conditions appear, and for simplicity we consider the homogeneous equation

$$\Delta\phi = 0 \tag{4.16}$$

with zero Cauchy data. Here we specify on T the values of

$$t\phi = \xi_p, \quad t*d\phi = \eta_{N-p-1}. \tag{4.17}$$

The number of components so specified is not in general equal to the number of components of ϕ, and indeed if $p = N$ (4.17) are empty conditions. However, we shall prove that the appropriate conditions of orthogonality are necessary and sufficient for the existence of a solution. The eigenforms satisfy

$$d\delta\psi = 0, \quad d\psi = 0,$$

with $t\psi = 0$ on T and any such ψ is an eigenform. This is demonstrated by showing that an eigenform is necessarily closed, as follows: $d\psi$ is harmonic, with zero Cauchy data, vanishing normal part on T, and with $t*d.d\psi = 0$. Hence $d\psi$ vanishes by the uniqueness in Theorem II.

The orthogonality conditions arise from the eigenforms ρ of the dual backward problem based on Σ. These satisfy

$$\delta d\rho = 0, \quad \delta\rho = 0, \tag{4.18}$$

with $t*\rho = 0$ on T, and with zero Cauchy data on Σ. The backward form of Theorem IV shows that infinitely many linearly independent eigenforms ρ exist for each surface Σ. Thus the orthogonality conditions

$$\int_{T_\Sigma} (\xi \wedge *d\rho - \rho \wedge \eta) = 0 \tag{4.19}$$

take effect.

THEOREM IX. *Let Cauchy data zero be assigned on S. Let the data in* (4.17) *satisfy compatibility conditions of orders* $k = 0$ *and* 1. *Then there exists a harmonic form satisfying* (4.17) *if and only if* (4.19) *holds.*

We have shown that (4.19) is necessary. To prove that it is sufficient, we begin by constructing a harmonic form ϕ_1 with zero Cauchy data and with $t*d\phi_1 = \eta$. By Theorem II such a form exists (with $t*\phi_1 = 0$) and is of class C^1. The orthogonality conditions apply to ϕ_1 so that

$$\int_{T_\Sigma} (\phi_1 \wedge *d\rho - \rho \wedge *d\phi_1) = 0,$$

and on subtraction we get

$$\int_{T_\Sigma} (\xi - t\phi_1) \wedge *d\rho = 0. \tag{4.20}$$

By Theorem IV, or its dual, we can, however, choose $t*d\rho$ to be an arbitrary closed form which vanishes on C_Σ. The lemma now shows that

$$d_T(\xi - t\phi_1) = 0. \tag{4.21}$$

Again making use of the dual of Theorem IV, we see that it is possible to construct a form χ with

$$d\delta\chi = 0, \quad d\chi = 0,$$

having zero Cauchy data and satisfying the boundary conditions

$$t\chi = \xi - t\phi_1, \quad t\delta\chi = 0.$$

The necessary condition dual to (3.8) is satisfied here in view of (4.21), while the remaining conditions are clearly fulfilled. Now we take the C^1 form

$$\phi = \phi_1 + \chi$$

and observe that $\Delta\phi = 0$, while

$$t\phi = t\phi_1 + \xi - t\phi_1 = \xi$$

and

$$t*d\phi = t*d\phi_1 + 0 = \eta.$$

This completes the proof.

As remarked above, these boundary conditions are empty if $p = N$. For $p = 0$, on the other hand, the orthogonality conditions are highly restrictive and determine either one of ξ and η in terms of the other. This set of boundary conditions has the property that its adjoint set, in the sense of the theory of

differential operators, is the dual set in the notation of generalized potential theory.

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SOME GENERATING FUNCTIONS FOR LAGUERRE AND HERMITE POLYNOMIALS

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1. Introduction. The Laguerre polynomials $L_n^{(\alpha)}(x)$ have the following hypergeometric and Rodrigues representation (3, pp. 188, 189):

$$L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!}\ {}_1F_1\left[\begin{matrix} -n; \\ 1+\alpha; \end{matrix} x\right], \tag{1}$$

$$L_n^{(\alpha)}(x) = \frac{e^x x^{-\alpha}}{n!}\ D_x^n[e^{-x} x^{n+\alpha}]. \tag{2}$$

The Hermite polynomials $H_n(x)$ have the representations (3, p. 193):

$$H_n(x) = (2x)^n\ {}_2F_0(-\tfrac{1}{2}n, -\tfrac{1}{2}n + \tfrac{1}{2}; -; -x^{-2}), \tag{3}$$

$$H_n(x) = (-1)^n e^{x^2} D_x^n e^{-x^2}. \tag{4}$$

This note will present several generating functions for the $L_n^{(\alpha)}(x)$ and $H_n(x)$. The result (5) was recently obtained, by an entirely different method, by Weisner (7). The other following results are believed to be new:

$$(1-t)^{-1-\alpha+a}(1-t+vt)^{-a} \exp\left(\frac{-xt}{1-t}\right) {}_1F_1\left[\begin{matrix} a; \\ 1+\alpha; \end{matrix} \frac{vtx}{(1-t)(1-t+vt)}\right] = \sum_{n=0}^{\infty} {}_2F_1\left[\begin{matrix} -n, a; \\ 1+\alpha; \end{matrix} v\right] t^n L_n^{(\alpha)}(x); \tag{5}$$

$$[1 + 2vt(x-t)]^{-a} \exp(2xt - t^2)\ {}_2F_0\left[\begin{matrix} \tfrac{1}{2}a, \tfrac{1}{2}a + \tfrac{1}{2}; \\ - \end{matrix} ; \frac{-4v^2t^2}{[1+2vt(x-t)]^2}\right] \sim \sum_{n=0}^{\infty} {}_2F_0\left[\begin{matrix} -n, a; \\ -; \end{matrix} v\right] \frac{t^n}{n!} H_n(x); \tag{6}$$

$$\tfrac{1}{2}\exp\left(\frac{-xt}{1-t}\right)(1-t)^{a-\alpha-1}\left\{(1-t-vt)^{-a}\ {}_1F_1\left[\begin{matrix} a; \\ 1+\alpha; \end{matrix} \frac{-tvx}{(1-t)(1-t-vt)}\right] + (1-t+vt)^{-a}\ {}_1F_1\left[\begin{matrix} a; \\ 1+\alpha; \end{matrix} \frac{tvx}{(1-t)(1-t+vt)}\right]\right\} = \sum_{n=0}^{\infty} {}_4F_2\left[\begin{matrix} -\tfrac{1}{2}n, -\tfrac{1}{2}n + \tfrac{1}{2}, \tfrac{1}{2}a, \tfrac{1}{2}a + \tfrac{1}{2}; \\ \tfrac{1}{2}, \tfrac{1}{2} + \tfrac{1}{2}\alpha, \quad 1 + \tfrac{1}{2}\alpha; \end{matrix} v^2\right] L_n^{(\alpha)}(x)\, t^n; \tag{7}$$

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$$(8)\quad \tfrac{1}{2}(1-t)^{-1-\alpha}\exp\left(\frac{-xt}{1-t}\right)\left\{\exp\left(\frac{2vt}{1-t}\right){}_0F_1\left[\begin{matrix}-;\\ 1+\alpha;\end{matrix}\ \frac{-2vxt}{(1-t)^2}\right]\right.$$
$$\left.+\exp\left(\frac{-2vt}{1-t}\right){}_0F_1\left[\begin{matrix}-;\\ 1+\alpha;\end{matrix}\ \frac{2vxt}{(1-t)^2}\right]\right\}$$
$$=\sum_{n=0}^{\infty}{}_2F_2\left[\begin{matrix}-\tfrac{1}{2}n,\ -\tfrac{1}{2}n+\tfrac{1}{2};\\ \tfrac{1}{2},\ \tfrac{1}{2}+\tfrac{1}{2}\alpha,\ 1+\tfrac{1}{2}\alpha;\end{matrix}\ v^2\right]L_n^{(\alpha)}(x)\,t^n;$$

$$(9)\quad \tfrac{1}{2}\exp(2xt-t^2)\left\{[1-vt(x-t)]^{-a}{}_2F_0\left[\begin{matrix}\tfrac{1}{2}a,\ \tfrac{1}{2}a+\tfrac{1}{2};\\ -;\end{matrix}\ \frac{-v^2t^2}{[1-vt(x-t)]^2}\right]\right.$$
$$\left.+[1+vt(x-t)]^{-a}{}_2F_0\left[\begin{matrix}\tfrac{1}{2}a,\ \tfrac{1}{2}a+\tfrac{1}{2};\\ -;\end{matrix}\ \frac{-v^2t^2}{[1+vt(x-t)]^2}\right]\right\}$$
$$\sim\sum_{n=0}^{\infty}{}_4F_1\left[\begin{matrix}-\tfrac{1}{2}n,\ -\tfrac{1}{2}n+\tfrac{1}{2},\ \tfrac{1}{2}a,\ \tfrac{1}{2}a+\tfrac{1}{2};\\ \tfrac{1}{2};\end{matrix}\ v^2\right]\frac{H_n(x)\,t^n}{n!};$$

$$(10)\quad \exp[2xt-t^2-v^2t^2]\cosh[2vt(x-t)]$$
$$=\sum_{n=0}^{\infty}{}_2F_1\left[\begin{matrix}-\tfrac{1}{2}n,\ -\tfrac{1}{2}n+\tfrac{1}{2};\\ \tfrac{1}{2};\end{matrix}\ v^2\right]\frac{H_n(x)\,t^n}{n!};$$

$$(11)\quad \exp(2xt-t^2)(1+ut^2)^{-a}\,{}_1F_1\left[\begin{matrix}a;\\ \tfrac{1}{2};\end{matrix}\ \frac{ut^2(x-t)^2}{1+ut^2}\right]$$
$$=\sum_{n=0}^{\infty}{}_2F_1\left[\begin{matrix}-\tfrac{1}{2}n,\ -\tfrac{1}{2}n+\tfrac{1}{2},\ a;\\ \tfrac{1}{2};\end{matrix}\ u\right]\frac{H_n(x)\,t^n}{n!}.$$

Equations (6) and (9) give divergent generating functions, but the rest are convergent for t sufficiently small. In equations (5), (6), (7), (9), (11), a is arbitrary. It should be noted that putting $a = -k$ $(k = 0, 1, 2, \ldots)$ in (5) and (7) reduces them to a form involving the general Laguerre polynomial itself, doing this in (11) reduces the left side to a form involving the Laguerre polynomial for the special case $\alpha = -\tfrac{1}{2}$, and doing this in (6) and (9) reduces the left sides to forms involving the Hermite polynomials. It should further be noted that the

$${}_2F_1\left[\begin{matrix}-\tfrac{1}{2}n,\ -\tfrac{1}{2}n+\tfrac{1}{2};\\ \tfrac{1}{2};\end{matrix}\ v^2\right].$$

on the right side of (10) are essentially Tchebycheff polynomials of the first kind in the argument $(1-v^2)^{-\frac{1}{2}}$.

2. Proof of (5). From (2) it follows that

$$(12)\qquad L_n^{(\alpha)}(x)=\frac{e^x x^{-\alpha}}{2\pi i}\int_C \frac{z^{n+\alpha}e^{-z}}{(z-x)^{n+1}}\,dz,$$

for $x \neq 0$, where C is a simple closed path about $z = x$, not containing $z = 0$. Multiply by

$$(13)\qquad t^n\,{}_{p+1}F_q\left[\begin{matrix}-n,\ c_1,\ c_2,\ \ldots,\ c_p;\\ d_1,\ d_2,\ \ldots,\ d_q;\end{matrix}\ v\right],$$

sum over n, and switch operations to find:

$$B(v,x,t) \equiv \sum_{n=0}^{\infty} {}_{p+1}F_q\left[\begin{matrix} -n, c_1, c_2, \ldots, c_p; \\ d_1, d_2, \ldots, d_q; \end{matrix} v\right] t^n L_n^{(\alpha)}(x) \tag{14}$$
$$\sim \frac{e^x x^{-\alpha}}{2\pi i} \int_C \frac{z^\alpha e^{-z}}{z-x} A(z)\, dz,$$

where

$$A(z) = \sum_{n=0}^{\infty} {}_{p+1}F_q\left[\begin{matrix} -n, c_1, \ldots, c_p; \\ d_1, \ldots, d_q; \end{matrix} v\right]\left(\frac{tz}{z-x}\right)^n. \tag{15}$$

Equation (15) can be summed directly or by a special case of a formula of Chaundy (**2**, p. 62) to give:

$$A(z) = \frac{z-x}{z-x-tz} {}_{p+1}F_q\left[\begin{matrix} 1, c_1, \ldots, c_p; \\ d_1, \ldots, d_q; \end{matrix} \frac{-vtz}{z-x-tz}\right]. \tag{16}$$

The question of interchange of operations in (14) must now be considered. The preliminary result (18) to be obtained below is general enough to contain both convergent and divergent cases. For $p > q$, the series in (16) diverges and the result is a purely formal one. For $p \leqslant q$, the series converges uniformly for a neighborhood of $t = 0$ and the interchange of operations in (14) is justified. In the latter case, replace the equivalence symbol $\sim$ in (14) by equals signs.

In the general case then,

$$B(v,x,t) \sim \frac{e^x x^{-\alpha}}{2\pi i} \int_C \frac{z^\alpha e^{-z}}{z-x-tz} {}_{p+1}F_q\left[\begin{matrix} 1, c_1, \ldots, c_p; \\ d_1, \ldots, d_q; \end{matrix} \frac{-vtz}{z-x-tz}\right] dz$$
$$\sim \frac{e^x x^{-\alpha}}{2\pi i} \int_C \frac{z^\alpha e^{-z}}{z-x-tz} \sum_{n=0}^{\infty} \frac{(c_1)_n \ldots (c_p)_n (-vtz)^n}{(d_1)_n \ldots (d_q)_n (z-x-tz)^n} dz \tag{17}$$
$$\sim e^x x^{-\alpha} \sum_{n=0}^{\infty} \frac{(c_1)_n \ldots (c_p)_n (-vt)^n}{(d_1)_n \ldots (d_q)_n n!} \frac{n!}{2\pi i} \int_C \frac{z^{\alpha+n} e^{-z}}{(z-x-tz)^{n+1}} dz.$$

In the last line, the same comments as to justification of the interchange of operations apply as before. The last line may be converted by (12) provided t is sufficiently small. This gives:

$$B(v,x,t) \sim e^x x^{-\alpha} \sum_{n=0}^{\infty} \frac{(c_1)_n \ldots (c_p)_n (-vt)^n}{(d_1)_n \ldots (d_q)_n n! (1-t)^{n+1}} [D_z^n (z^{\alpha+n} e^{-z})]_{z=x/(1-t)}$$
$$\sim \frac{e^x x^{-\alpha}}{(1-t)} \sum_{n=0}^{\infty} \frac{(c_1)_n \ldots (c_p)_n (-vt)^n}{(d_1)_n \ldots (d_q)_n (1-t)^n n!} \left[\frac{n! L_n^{(\alpha)}(z)}{e^z z^{-\alpha}}\right]_{z=x/(1-t)} \tag{18}$$
$$\sim (1-t)^{-1-\alpha} \exp\left(\frac{-xt}{1-t}\right) \sum_{n=0}^{\infty} \frac{(c_1)_n \ldots (c_p)_n (-vt)^n}{(d_1)_n \ldots (d_q)_n (1-t)^n} L_n^{(\alpha)}[x/(1-t)].$$

Result (5) now follows by taking

$$p = 1, q = 1, c_1 = a, d_1 = 1 + \alpha \tag{19}$$

and using the result **(2**, p. 62, quoted in **1**, p. 949**)**:

$$(20)\qquad (1-t)^{-a}\,{}_1F_1\left[\begin{matrix} a \\ 1+\alpha \end{matrix};\frac{-xt}{1-t}\right]=\sum_{n=0}^{\infty}\frac{(a)_n L_n^{(\alpha)}(x)\,t^n}{(1+\alpha)_n}.$$

Another previously known result **(6**, p. 98, eqn. (5.1.15)**)** also follows from (18) by taking

$$(21)\qquad p=0,\ q=1,\ d_1=1+\alpha$$

and using the result **(6**, p. 98**)**

$$(22)\qquad e^t\,{}_0F_1(-;1+\alpha;-xt)=\sum_{n=0}^{\infty}\frac{L_n^{(\alpha)}(x)\,t^n}{(1+\alpha)_n}.$$

3. Proof of (6). The development of (6) follows a pattern similar to that of §2. From (4) it follows that

$$(23)\qquad H_n(x)=(-1)^n\frac{e^{x^2}n!}{2\pi i}\int_C\frac{e^{-z^2}\,dz}{(z-x)^{n+1}}.$$

Multiply (23) by

$$(24)\qquad \frac{t^n}{n!}\,{}_{p+1}F_q\left[\begin{matrix} -n,\ c_1,\ \dots,\ c_p; \\ d_1,\ \dots,\ d_q; \end{matrix}v\right],$$

and proceed in a manner exactly like that of §2 to find, after two interchanges of operations:

$$\begin{aligned}(25)\quad D(v,x,t)&\equiv\sum_{n=0}^{\infty}{}_{p+1}F_q\left[\begin{matrix} -n,\ c_1,\ \dots,\ c_p; \\ d_1,\ \dots,\ d_q; \end{matrix}v\right]\frac{t^n H_n(x)}{n!}\\ &\sim e^{x^2}\sum_{n=0}^{\infty}\frac{(c_1)_n\dots(c_p)_n\,(vt)^n}{(d_1)_n\dots(d_q)_n\,n!}\,\frac{n!}{2\pi i}\int_C\frac{e^{-z^2}\,dz}{(z-x+t)^{n+1}}\\ &\sim e^{x^2}\sum_{n=0}^{\infty}\frac{(c_1)_n\dots(c_p)_n\,(vt)^n}{(d_1)_n\dots(d_q)_n\,n!}\,[(-1)^n e^{-z^2}H_n(z)]_{z=x-t}\\ &\sim\exp(2xt-t^2)\sum_{n=0}^{\infty}\frac{(c_1)_n\dots(c_p)_n\,(-vt)^n H_n(x-t)}{(d_1)_n\dots(d_q)_n\,n!}.\end{aligned}$$

Result (6) follows from (25) by taking

$$(26)\qquad p=1,\ c_1=a,\ q=0,$$

and using the formula **(1**, p. 948**)**:

$$(27)\qquad (1-2xt)^{-a}\,{}_2F_0\left[\begin{matrix} \tfrac{1}{2}a,\ \tfrac{1}{2}a+\tfrac{1}{2}; \\ -; \end{matrix}\frac{-4t^2}{(1-2xt)^2}\right]\sim\sum_{n=0}^{\infty}\frac{(a)_n H_n(x)\,t^n}{n!}.$$

4. Proof of (7). The developments in §§3 and 4 depended on Chaundy's summation formula, which was used, for example, to convert (15) into (16). The whole procedure may be repeated to yield new results if one will instead use **(1**, p. 947**)**:

$$(28)\qquad \begin{aligned}&(1-t)^{-1}\,{}_{p+2}F_q\left[\begin{matrix}\tfrac{1}{2},\,1,\,\alpha_1,\ldots,\alpha_p;\\ \beta_1,\ldots,\beta_q;\end{matrix}\ \frac{xt^2}{(1-t)^2}\right]\\ &\sim\sum_{n=0}^{\infty} t^n\,{}_{p+2}F_q\left[\begin{matrix}-\tfrac{1}{2}n,\,-\tfrac{1}{2}n+\tfrac{1}{2},\,\alpha_1,\ldots,\alpha_p;\\ \beta_1,\ldots,\beta_q;\end{matrix}\ x\right].\end{aligned}$$

Start again with (12). Then it follows that

$$(29)\qquad \begin{aligned}E(v,x,t)&\equiv\sum_{n=0}^{\infty}{}_{p+2}F_q\left[\begin{matrix}-\tfrac{1}{2}n,\,-\tfrac{1}{2}n+\tfrac{1}{2},\,a_1,\ldots,a_p;\\ b_1,\ldots,b_q;\end{matrix}\ v^2\right]L_n^{(\alpha)}(x)\,t^n\\ &\sim\frac{e^x x^{-\alpha}}{2\pi i}\int_C\frac{e^{-z}z^{\alpha}}{z-x}A(z)\,dz,\end{aligned}$$

where

$$(30)\qquad A(z)=\sum_{n=0}^{\infty}\left(\frac{zt}{z-x}\right)^n{}_{p+2}F_q\left[\begin{matrix}-\tfrac{1}{2}n,\,-\tfrac{1}{2}n+\tfrac{1}{2},\,a_1,\ldots,a_p;\\ b_1,\ldots,b_q;\end{matrix}\ v^2\right].$$

The interchange of operations gives valid convergent results if $p+1\leqslant q$, and a formal divergent result if $p+1>q$. The application of (28) to (30) and the substitution of the result in (29) yields:

(31) $E(v,x,t)$

$$\sim\frac{e^x x^{-\alpha}}{2\pi i}\int_C\frac{e^{-z}z^{-\alpha}}{z-x-zt}\,{}_{p+2}F_q\left[\begin{matrix}\tfrac{1}{2},\,1,\,a_1,\ldots,a_p;\\ b_1,\ldots,b_q;\end{matrix}\ \frac{v^2z^2t^2}{(z-x-zt)^2}\right]dz.$$

Interchange operations again to find:

(32) $E(v,x,t)$

$$\sim(1-t)^{-1-\alpha}\exp\left(-\frac{xt}{1-t}\right)\sum_{n=0}^{\infty}\frac{(\tfrac{1}{2})_n\,(a_1)_n\ldots(a_p)_n\,(vt)^{2n}}{(b_1)_n\ldots(b_q)_n\,(1-t)^{2n}}\cdot F,$$

where

$$(33)\qquad F=\frac{x^{-\alpha}(1-t)^{\alpha}\exp[x/(1-t)]}{2\pi i}\int_C\frac{e^{-z}z^{\alpha+2n}\,dz}{\left[z-x-\dfrac{xt}{1-t}\right]^{2n+1}}.$$

For t sufficiently small, $xt/(1-t)$ lies within C and hence

$$(34)\qquad F=L_{2n}^{(\alpha)}[x/(1-t)].$$

So far then, the result is:

(35) $E(v,x,t)$

$$\sim(1-t)^{-1-\alpha}\exp\left(\frac{-xt}{1-t}\right)\sum_{n=0}^{\infty}\frac{(\tfrac{1}{2})_n\,(a_1)_n\ldots(a_p)_n\,(vt)^{2n}}{(b_1)_n\ldots(b_q)_n\,(1-t)^{2n}}L_{2n}^{(\alpha)}\left(\frac{x}{1-t}\right).$$

As a special case of (35), take

(36) $p=2,\ q=3,\ a_1=\tfrac{1}{2}a,\ a_2=\tfrac{1}{2}a+\tfrac{1}{2},\ b_1=\tfrac{1}{2},\ b_2=\tfrac{1}{2}+\tfrac{1}{2}\alpha,\ b_3=1+\tfrac{1}{2}\alpha,$

and recall that:

$$(2c)_{2n} = 2^{2n}\,(c)_n\,(c+\tfrac{1}{2})_n. \tag{37}$$

Then (35) becomes:

$$\sum_{n=0}^{\infty} {}_4F_3\left[\begin{matrix} -\frac{1}{2}n, -\frac{1}{2}n+\frac{1}{2}, \frac{1}{2}a, \frac{1}{2}a+\frac{1}{2}; \\ \frac{1}{2}, \frac{1}{2}+\frac{1}{2}\alpha, 1+\frac{1}{2}\alpha: \end{matrix} v^2\right] L_n^{(\alpha)}(x)\,t^n$$
$$= (1-t)^{-1-\alpha} \exp\left(-\frac{xt}{1-t}\right) \sum_{n=0}^{\infty} \frac{(a)_{2n}\,(vt)^{2n}}{(1+\alpha)_{2n}\,(1-t)^{2n}} L_{2n}^{(\alpha)}\left(\frac{x}{1-t}\right). \tag{38}$$

From the known result (20), it easily follows that

$$(1-t)^{-a}\,{}_1F_1\left[\begin{matrix} a \\ 1+\alpha; \end{matrix} \frac{-xt}{1-t}\right] + (1+t)^{-a}\,{}_1F_1\left[\begin{matrix} a \\ 1+\alpha; \end{matrix} \frac{xt}{1+t}\right]$$
$$= 2\sum_{n=0}^{\infty} \frac{(a)_{2n}\,L_{2n}^{(\alpha)}(x)\,t^{2n}}{(1+\alpha)_{2n}}. \tag{39}$$

The application of (39) to (38) yields (7).

5. Proof of (8). As a special case of (35), take

$$p = 0, q = 3, b_1 = \tfrac{1}{2}, b_2 = \tfrac{1}{2}+\tfrac{1}{2}\alpha, b_3 = 1+\tfrac{1}{2}\alpha. \tag{40}$$

From the known result

$$e^t\,{}_0F_1(-; 1+\alpha; -xt) = \sum_{n=0}^{\infty} \frac{t^n L_n^{(\alpha)}(x)}{(1+\alpha)_n}, \tag{41}$$

it is easy to get:

$$e^t\,{}_0F_1(-; 1+\alpha; -xt) + e^{-t}\,{}_0F_1(-; 1+\alpha; xt)$$
$$= 2\sum_{n=0}^{\infty} \frac{t^{2n} L_{2n}^{(\alpha)}(x)}{(1+\alpha)_{2n}}. \tag{42}$$

The application of (42) to the special case represented by (40) gives (8).

6. Proofs of (9), (10), (11). Start from (23), multiply by

$$\frac{t^n}{n!}\,{}_{p+2}F_q\left[\begin{matrix} -\frac{1}{2}n, -\frac{1}{2}n+\frac{1}{2}, a_1, \ldots, a_p; \\ b_1, \ldots, b_q; \end{matrix} v^2\right], \tag{43}$$

sum on n, and interchange operations exactly as done previously. The result, stated without intermediate steps is:

$$\sum_{n=0}^{\infty} {}_{p+2}F_q\left[\begin{matrix} -\frac{1}{2}n, -\frac{1}{2}n+\frac{1}{2}, a_1, \ldots, a_p; \\ b_1, \ldots, b_q; \end{matrix} v^2\right] \frac{H_n(x)\,t^n}{n!}$$
$$\sim \exp(2xt - t^2) \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n\,(\frac{1}{2}vt)^{2n}}{(b_1)_n \ldots (b_q)_n\,n!} H_{2n}(x-t). \tag{44}$$

In (44), take

$$p = 2, q = 1, a_1 = \tfrac{1}{2}a, a_2 = \tfrac{1}{2}a+\tfrac{1}{2}, b_1 = \tfrac{1}{2}. \tag{45}$$

From (27), obtain

$$(46)\qquad (1-2xt)^{-a}\,{}_2F_0\left[\begin{matrix}\frac{1}{2}a,\ \frac{1}{2}a+\frac{1}{2};\\ -;\end{matrix}\ \frac{-4t^2}{(1-2xt)^2}\right]$$
$$+(1+2xt)^{-a}\,{}_2F_0\left[\begin{matrix}\frac{1}{2}a,\ \frac{1}{2}a+\frac{1}{2};\\ -;\end{matrix}\ \frac{-4t^2}{(1+2xt)^2}\right]$$
$$\sim 2\sum_{n=0}^{\infty}\frac{(a)_{2n}H_{2n}(x)\,t^{2n}}{(2n)!}.$$

The combination of (44), (45), and (46) will yield (9).

As the next special case of (44), take

$$(47)\qquad p=0,\ q=1,\ b_1=\tfrac{1}{2}.$$

From **(6**, p. 102**)**

$$\exp(2xt-t^2)=\sum_{n=0}^{\infty}\frac{H_n(x)\,t^n}{n!}$$

obtain:

$$(48)\qquad e^{-t^2}\cosh 2xt=\sum_{n=0}^{\infty}\frac{H_{2n}(x)\,t^{2n}}{(2n)!}.$$

The combination (44), (47), and (48) will yield (10).

For the last special case of (44), take

$$(49)\qquad p=1,\ q=1,\ a_1=a,\ b_1=\tfrac{1}{2}.$$

Substitute the result **(6**, p. 102**)**:

$$(50)\qquad H_{2n}(y)=(-1)^n\,2^{2n}\,n!\,L_n^{(-\frac{1}{2})}(y^2).$$

So far, (49) and (50) give, for $v^2=u$:

$$(51)\qquad \sum_{n=0}^{\infty}{}_3F_1\left[\begin{matrix}-\frac{1}{2}n,\ -\frac{1}{2}n+\frac{1}{2},\ a;\\ \frac{1}{2};\end{matrix}\ u\right]\frac{H_n(x)\,t^n}{n!}$$
$$=\exp(2xt-t^2)\sum_{n=0}^{\infty}\frac{(a)_n\,(-ut^2)^n}{(\frac{1}{2})_n}\,L_n^{(-\frac{1}{2})}[(x-t)^2].$$

The application of (20) now yields (11).

7. More general results. Make the definition

$$(52)\quad f_n\left[k;\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix}\ x\right]$$
$$={}_{k+p}F_q\left[\begin{matrix}\frac{-n}{k},\ \frac{-n+1}{k},\ \ldots,\ \frac{-n+k-1}{k},\ \alpha_1,\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix}\ x\right],$$

where $k=1,2,3,\ldots$.

Then by direct summation and use of the relation

$$(54)\qquad (c)_{kj} = k^{kj}\,(c/k)_j\,((c+1)/k)_j \cdots ((c+k-1)/k)_j$$

it follows that

$$(55)\quad (1-t)^{-a}\,{}_{k+p}F_q\left[\begin{matrix}\frac{a}{k}, \frac{a+1}{k}, \ldots, \frac{a+k-1}{k}, \alpha_1, \alpha_2, \ldots, \alpha_p; \\ \beta_1, \beta_2, \ldots, \beta_q;\end{matrix} x\left(\frac{-t}{1-t}\right)^k\right] \sim \sum_{n=0}^{\infty} \frac{(a)_n t^n}{n!} f_n\left[k; \begin{matrix}\alpha_1, \alpha_2, \ldots, \alpha_p; \\ \beta_1, \beta_2, \ldots, \beta_q;\end{matrix} x\right].$$

Equation (55) is the obvious generalization of (**1**, p. 947, eqns. 24 and 25). For $a = 1$, (55) gives a generalization of (28), which is the special case $k = 2$ of (55).

It is thus possible to generalize the procedures of the above sections. The results are so general and so complicated, it does not seem that they would be of much use. Just one of the many possible is presented here for illustration. Let w_k be a kth root of unity, other than 1 itself. Then:

$$(56)\quad k^{-1}(1-t)^{-1-\alpha}\exp\left(-\frac{xt}{1-t}\right)\sum_{i=0}^{k-1}\left(1+\frac{w_k^i tv}{1-t}\right)^{-a} {}_1F_1\left[\begin{matrix}a \\ 1+\alpha;\end{matrix} \frac{w_k^i tvx}{(1-t)(1-t+w_k^i tv)}\right] \sim \sum_{n=0}^{\infty} t^n f_n\left[k; \begin{matrix}\frac{a}{k}, \frac{a+1}{k}, \ldots, \frac{a+k-1}{k}; \\ \frac{1}{k}, \frac{2}{k}, \ldots, \frac{k-1}{k}, \frac{1+\alpha}{k}, \frac{2+\alpha}{k}, \ldots, \frac{k+\alpha}{k};\end{matrix} v^k\right] L_n^{(\alpha)}(x).$$

For $k = 2$, (56) reduces to (7) and for $k = 1$, (56) reduces to (5). The interested reader will be able to find similar generalizations of (6) through (11).

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SOME POLYNOMIALS OF TOUCHARD CONNECTED WITH THE BERNOULLI NUMBERS

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In a recent paper **(4)** Touchard has constructed a set of polynomials $\Omega_n(z)$ such that

$$B^r\,\Omega_n(B) = \begin{cases} 0 & (0 \leqslant r < n) \\ K_n & (r = n), \end{cases} \tag{1}$$

where after expansion of the left member B^m is replaced by B_m,

$$e^{Bz} = \sum_{m=0}^{\infty} B_m x^m/m! = \frac{x}{e^x - 1},$$

and

$$K_n = \frac{(-1)^n}{2n+1}\,\frac{1}{2^n}\,\frac{(n!)^4}{[1.3.5\ldots(2n-1)]^2}. \tag{2}$$

(Touchard writes $Q_n(z)$ in place of $\Omega_n(z)$; we have changed the notation in order to avoid a clash with the Legendre function of the second kind.) It is proved by Touchard that

$$\Omega_{n+1}(z) = (2z+1)\,\Omega_n(z) + \frac{n^4}{4n^2-1}\,\Omega_{n-1}(z). \tag{3}$$

Using (3), Wyman and Moser **(5)** showed that

$$\Omega_n(z) = 2^n n! \binom{2n}{n}^{-1} \sum_{2r \leqslant n} \binom{2z+n-2r}{n-2r}\binom{z}{r}^2. \tag{4}$$

In the usual notation of generalized hypergeometric functions (4) may be written

$$\Omega_n(z) = 2^n n! \binom{2n}{n}^{-1}\binom{2z+n}{n} \cdot {}_4F_3\left[\begin{matrix} -\frac{n}{2}, \frac{1}{2}-\frac{n}{2}, -z, -z; \\ 1, -z-\frac{n}{2}, -z+\frac{1}{2}-\frac{n}{2} \end{matrix}\right]. \tag{5}$$

Now Bateman **(1; 2)** has introduced a polynomial

$$F_n(z) = {}_3F_2\left[\begin{matrix} -n, n+1, \frac{1}{2}(1+z); \\ 1, 1 \end{matrix}\right] \tag{6}$$

such that

$$F_n(-z) = (-1)^n F_n(z); \tag{7}$$

also in place of (6) there is the alternate expansion

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$$(8) \qquad F_n(z) = (-1)^n \binom{z}{n} {}_4F_3\left[\begin{matrix} -\frac{n}{2}, \frac{1}{2}-\frac{n}{2}, \frac{1+z}{2}, \frac{1+z}{2}; \\ 1, \frac{z+1-n}{2}, \frac{z+2-n}{2} \end{matrix}\right].$$

Using (7), (8) becomes

$$(9) \qquad F_n(2z+1) = (-1)^n \binom{2z+n}{n} {}_4F_3\left[\begin{matrix} -\frac{n}{2}, \frac{1}{2}-\frac{n}{2}, -z, -z; \\ 1, -z-\frac{n}{2}, -z+\frac{1}{2}-\frac{n}{2} \end{matrix}\right].$$

Comparison of (9) with (5) yields at once

$$(10) \qquad \Omega_n(z) = (-1)^n 2^n n! \binom{2n}{n}^{-1} F_n(2z+1).$$

In the next place we recall Bateman's formula **(2)**

$$(11) \qquad F_n\left(\frac{d}{dz}\right) z \operatorname{cosech} z = \operatorname{cosech} z . Q_n(\coth z),$$

where $Q_n(z)$ denotes the Legendre function of the second kind. We have also in the notation of Nörlund **(3**, Ch. 2**)**

$$z \operatorname{cosech} z = \sum_0^\infty D_m z^m/m! \qquad (D_{2m+1} = 0),$$

where (symbolically)

$$(12) \qquad D_m = (2B+1)^m.$$

Expanding the left member of (11) we get

$$\sum_{r=0}^\infty \frac{z^r}{r!} D^r F_n(D);$$

in view of (10) and (12) this is equal to

$$(-1)^n (2^n n!)^{-1} \binom{2n}{n} \sum_{r=0}^\infty \frac{z^r}{r!} (2B+1)^r \Omega_n(B).$$

Since

$$Q_n(z) = \frac{2^n (n!)^2}{(2n+1)!} \frac{1}{z^{n+1}} F\left(\frac{n}{2}+\frac{1}{2}, \frac{n}{2}+1; \frac{n}{2}+\frac{3}{2}; z^{-2}\right) \qquad (|z| > 1)$$

we accordingly get

$$(13) \qquad \sum_{r=0}^\infty \frac{z^r}{r!} (2B+1)^r \Omega_n(B)$$

$$= (-1)^n \frac{2^{2n} (n!)^5}{(2n+1)! (2n)!} \sinh^n z \operatorname{sech}^{n+1} z F\left(\frac{n}{2}+\frac{1}{2}, \frac{n}{2}+1; n+\frac{3}{2}; \tanh^2 z\right).$$

From (13) it is clear that

$$(2B+1)^r \Omega_n(B) = 0 \qquad (0 \leqslant r < n),$$

and therefore

$$B^r\,\Omega_n(B) = 0 \qquad (0 \leqq r < n). \tag{14}$$

As for $r = n$, we have

$$(2B+1)^n\,\Omega_n(B) = (-1)^n\,\frac{2^{2n}(n!)^6}{(2n+1)!\,(2n)!};$$

using (14) this becomes

$$B^n\,\Omega_n(B) = (-1)^n\,\frac{(n!)^4}{2^n(2n+1)[1.3.5\ldots(2n-1)]^2} = K_n \tag{15}$$

This evidently completes the proof of (1).

It may be of interest to remark that (1) can be *verified* rapidly in the following way. Using (6) and (10) we see that

$$\begin{aligned}\Omega_n(B) &= (-1)^n\,2^n\,n!\binom{2n}{n}^{-1}{}_3F_2\left[\begin{matrix}-n,\ n+1,\ B+1;\\ 1,\ 1\end{matrix}\right]\\ &= (-1)^n\,2n!\binom{2n}{n}^{-1}\sum_{s=0}^{n}(-1)^s\binom{n}{s}\binom{n+s}{s}\binom{B+s}{s}.\end{aligned}$$

But (3, p. 149)

$$(B+1)(B+2)\ldots(B+s) = \frac{s!}{s+1},$$

so that

$$\begin{aligned}\sum_{s=0}^{n}(-1)^s\binom{n}{s}\binom{n+s}{s}\binom{B+s}{s} &= \sum_{s=0}^{n}(-1)^s\binom{n}{s}\binom{n+s}{s}\frac{1}{s+1}\\ &= F(-n,\ n+1;\ 2;\ 1) = 0.\end{aligned}$$

Thus $\Omega_n(B) = 0$. Repeated use of the recurrence (3) now completes the proof of (14). Finally (3) yields

$$\begin{aligned}(2B+1)^n\,\Omega_n(B) &= (-1)^n\,\frac{n^4(n-1)^4\ldots1^4}{(4n^2-1)(4(n-1)^2-1)\ldots(4-1)}\\ &= (-1)^n\,\frac{(n!)^4}{(2n+1)[(2n-1)\ldots3.1]^2},\end{aligned}$$

which gives (15).

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ON TOUCHARD POLYNOMIALS

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1. Introduction. Wyman and Moser **(5)** have recently given an explicit form for some polynomials of Touchard **(4)**. Equation (6) of this note will give a simpler expression for these polynomials. Subsequently, generating functions (9) and (12) will be obtained.

2. A new expression. The expression given by Wyman and Moser in **(5)** is

$$Q_n(x) = 2^n n! \binom{2n}{n}^{-1} \sum_{r=0}^{[\frac{1}{2}n]} \binom{2x+n-2r}{n-2r} \binom{x}{r}^2. \tag{1}$$

If (1) is written in generalized hypergeometric form it becomes

$$Q_n(x) = \frac{n!\,(2x+1)_n}{(\frac{1}{2})_n\, 2^n}\, {}_4F_2\left[\begin{matrix} -\frac{1}{2}n,\ \frac{1}{2}(-n+1),\ -x,\ -x; \\ 1,\ -x-\frac{1}{2}n,\ -x+\frac{1}{2}-\frac{1}{2}n; \end{matrix} 1\right]. \tag{2}$$

For the time being, restrict x to the values

$$x = m, \qquad m = 0, 1, 2, \ldots, \tag{3}$$

and use the transformation formula from Bailey **(1**, p. 85, eqn. (3)**)**:

$$\begin{aligned} {}_3F_2\left[\begin{matrix} -n,\ -n,\ -m; \\ \frac{1}{2},\ -n-m; \end{matrix} 1\right] &= \frac{(1+\frac{1}{2}n)_m\,(\frac{1}{2}+\frac{1}{2}n)_m}{(\frac{1}{2})_m\,(1+n)_m}\, {}_4F_2\left[\begin{matrix} -\frac{1}{2}n,\ \frac{1}{2}(-n+1),\ -m,\ -m; \\ 1,\ -m-\frac{1}{2}n,\ -m+\frac{1}{2}-\frac{1}{2}n; \end{matrix} 1\right]. \end{aligned} \tag{4}$$

Then

$$\begin{aligned} Q_n(m) &= \frac{n!\,(2m+1)_n\,(\frac{1}{2})_m\,(1+n)_m}{(\frac{1}{2})_n\,2^n(1+\frac{1}{2}n)_m\,(\frac{1}{2}+\frac{1}{2}n)_m}\, {}_3F_2\left[\begin{matrix} -n,\ -n,\ -m; \\ 1,\ -n-m; \end{matrix} 1\right] \\ &= \frac{(1+m)_n\,n!}{(\frac{1}{2})_n\,2^n}\, {}_3F_2\left[\begin{matrix} -n,\ -n,\ -m; \\ 1,\ -n-m; \end{matrix} 1\right]. \end{aligned} \tag{5}$$

Thus it has been shown that the $Q_n(x)$ of (2) reduces to

$$Q_n(x) = \frac{(1+x)_n\,n!}{(\frac{1}{2})_n\,2^n}\, {}_3F_2\left[\begin{matrix} -n,\ -n,\ -x; \\ 1,\ -n-x; \end{matrix} 1\right] \tag{6}$$

for the special case of x any non-negative integer. But then, for fixed n, (1) and (6) (in the form (7) below) give two polynomials of degree n in x which coincide at an infinite number of values. Hence (2) and (6) are identical and

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(6) is a valid representation for general x (except for negative integers, in which case (6) is indeterminate).

By the many transformations possible on a ${}_3F_2$ of unit argument, equation (6) can be converted to other forms.

Equation (6) here is equivalent to the result obtained independently by Carlitz in equation (10) of the immediately preceding work **(2)**. This equivalence may be demonstrated by the formula given by Bailey **(1**, p. 22**)**:

$$\Gamma(\alpha_{123})\ \Gamma(\alpha_{124})\ \Gamma(\alpha_{128})\ Fp(0;45) = (-1)^n\ \Gamma(\alpha_{128})\ \Gamma(\alpha_{028})\ \Gamma(\alpha_{018})\ Fn(5;02).$$

Carlitz has shown that the $Q_n(x)$ are essentially the Bateman polynomials $F_n(2x+1)$. Hence the results to be obtained below may also be interpreted as results on the F_n.

3. Some more generating functions. The form (6) makes available generating functions in addition to the one given by Wyman and Moser in **(5)**. Convert (6) into:

$$Q_n(x) = \frac{n!\,n!\,n!}{(\frac{1}{2})_n\,2^n}\sum_{k=0}^{n}\frac{(1+x)_{n-k}\,(-x)_k\,(-1)^k}{(n-k)!\,(n-k)!\,k!\,k!}. \tag{7}$$

Then

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{Q_n(x)\,(\frac{1}{2})_n\,(2t)^n}{n!\,n!\,n!} &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(1+x)_{n-k}\,(-x)_k\,(-1)^k\,t^n}{(n-k)!\,(n-k)!\,k!\,k!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(1+x)_n\,(-x)_k\,(-1)^k\,t^{n+k}}{n!\,n!\,k!\,k!}\\ &= {}_1F_1(1+x;1;t)\cdot{}_1F_1(-x;1;-t).\end{aligned} \tag{8}$$

The last line may be converted by Kummer's first formula to get the generating function:

$$e^t[{}_1F_1(-x;1;-t)]^2 = \sum_{n=0}^{\infty}\frac{Q_n(x)\,(\frac{1}{2})_n\,(2t)^n}{n!\,n!\,n!}. \tag{9}$$

It should be noted that for x a non-negative integer, the ${}_1F_1$ on the left of (9) becomes a Laguerre polynomial, so that (9) is:

$$e^t[L_x(-t)]^2 = \sum_{n=0}^{\infty}\frac{Q_n(x)\,(\frac{1}{2})_n\,(2t)^n}{n!\,n!\,n!}. \tag{10}$$

For the next generating function, sum (7) as indicated below to get:

$$\sum_{n=0}^{\infty}\frac{Q_n(x)\,(\frac{1}{2})_n\,(2t)^n}{n!\,n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(1+x)_{n-k}\,(-x)_k\,(-1)^k\,n!\,t^n}{(n-k)!\,(n-k)!\,k!\,k!} \tag{11}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(1+x)_n\,(-x)_k\,(-1)^k\,(n+k)!\,t^{n+k}}{n!\,n!\,k!\,k!}$$

$$= \sum_{n=0}^{\infty} \frac{(1+x)_n\,t^n}{n!} \sum_{k=0}^{\infty} \frac{(-x)_k\,(-t)^k\,(1+n)_k}{k!\,k!}$$

$$= \sum_{n=0}^{\infty} \frac{(1+x)_n\,t^n}{n!}\,{}_2F_1\left[\begin{matrix}1+n,\ -x;\\ 1;\end{matrix}\ -t\right]$$

$$= (1+t)^x \sum_{n=0}^{\infty} \frac{(1+x)_n\,t^n}{n!}\,{}_2F_1\left[\begin{matrix}-n,\ -x;\\ 1;\end{matrix}\ \frac{t}{t+1}\right].$$

By a well-known formula due to Chaundy (3, p. 62) the last line above sums to give the generating function:

$$(1+t)^x\,(1-t)^{-1-x}\,{}_2F_1\left[\begin{matrix}-x,\ 1+x;\\ 1;\end{matrix}\ \frac{-t^2}{1-t^2}\right] = \sum_{n=0}^{\infty} \frac{Q_n(x)\,(\tfrac{1}{2})_n\,(2t)^n}{n!\,n!}. \tag{12}$$

The ${}_2F_1$ in (12) is essentially a Legendre polynomial if x is a nonnegative integer, in which case (12) becomes

$$(1+t)^x\,(1-t)^{-1-x}\,P_x\left(\frac{1+t^2}{1-t^2}\right) = \sum_{n=0}^{\infty} \frac{Q_n(x)\,(\tfrac{1}{2})_n\,(2t)^n}{n!\,n!}. \tag{13}$$

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ASYMPTOTIC EXPANSIONS II

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1. Introduction. In a previous paper **(1)** the authors considered the problem of finding an asymptotic formula for numbers or functions $B_{n,m}$ whose generating function is of the form

$$\exp(P_m(x)) = \sum_{n=0}^{\infty} B_{n,m} \frac{x^n}{n!}, \tag{1.1}$$

where $P_m(x)$ is a polynomial of degree m in x given by

$$P_m(x) = \sum_{k=1}^{m} a_k x^k, \qquad a_m \neq 0. \tag{1.2}$$

The above-mentioned paper contained the restriction that $a_k \geq 0$. In our present paper we remove this restriction and allow the coefficients a_k to be positive, negative or zero. However we do retain $a_m \neq 0$. In **(1)** it was shown that there is no loss in generality in assuming that the greatest common divisor of the values of k, for which $a_k \neq 0$, is one. This assumption we shall also retain throughout the present paper.

Since the degree m of the polynomial $P_m(x)$ is fixed we shall, wherever possible, suppress m from our notation. Iterated exponential functions occur throughout the paper. For this reason we shall use alternative notations e^x or $\exp(x)$ to denote the exponential function. In this notation we write (1.1) and (1.2) as

$$\exp(P(x)) = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \tag{1.3}$$

$$P(x) = \sum_{k=1}^{m} a_k x^k, \qquad a_m \neq 0. \tag{1.4}$$

2. Trigonometric polynomials. In our previous paper **(1)** the trigonometric polynomial $S(R, \theta)$ associated with $P(x)$ by means of

$$S(R, \theta) = \Re[P(Re^{i\theta})] = \sum_{k=1}^{m} a_k R^k \cos k\theta \tag{2.1}$$

played a very important role. We shall call this function the dominant function of $P(x)$ and throughout the paper R shall be considered as large and positive.

When the coefficients $a_k \geq 0$ the dominant function has a greatest maximum at $\theta = 0$. Further in a sense, which we shall explain later in the paper, this greatest maximum is unique in the range $0 \leq \theta \leq \pi$. However when we do not restrict the sign of the a_k the greatest maximum will, in general, occur at

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some value of θ which depends on R. There may also be other maxima which we shall define to be equivalent to the greatest maximum. From the point of view of finding an asymptotic formula for the B_n these equivalent maxima are important. In fact the asymptotic formula is obtained by expansions about the angles which give the greatest maximum and its equivalents.

For the above reasons we shall, in this section, discuss and classify the maxima of the dominant function $S(R, \theta)$. Throughout the section we restrict θ to be in the range $0 \leqslant \theta \leqslant \pi$.

Denoting differentiation with respect to θ by a prime then the values of θ which make $S(R, \theta)$ a maximum are contained in the solutions of the equation

$$S'(R, \theta) = -\sum_{k=1}^{m} k\, a_k\, R^k \sin k\theta = 0. \tag{2.2}$$

For a large positive value of R it is easily shown that all of the solutions of (2.2) for θ must have the form

$$\theta_r(R) = \frac{r\pi}{m} + \sum_{s=1}^{\infty} b_{s,r}\, R^{-s}, \tag{2.3}$$

where r is an integer in the range $0 \leqslant r \leqslant m$ and $b_{s,r}$ are constants.

The only values of r for which θ_r will give a maximum of $S(R, \theta)$ are those values that will satisfy the equation

$$a_m \cos(r\pi) = |a_m|, \tag{2.4}$$

which of course implies that consecutive maxima are separated by an angular displacement of approximately $2\pi/m$.

Definition 2.1 The angles θ_r of (2.3) which yield maxima of $S(R, \theta)$ shall be called the *asymptotic angles* of $S(R, \theta)$. The reason for the name will become apparent later in the paper.

Definition 2.2 Let θ_i, θ_j be two asymptotic angles of $S(R, \theta)$. These shall be called *equivalent* asymptotic angles if and only if

$$\lim_{R\to\infty} (S(R, \theta_i) - S(R, \theta_j)) = \text{finite constant}. \tag{2.5}$$

Definition 2.3 The maxima associated with equivalent asymptotic angles are called *equivalent* maxima.

THEOREM 2.1. *There is no other asymptotic angle equivalent to $\theta = 0$ when $\theta = 0$ is an asymptotic angle of $S(R, \theta)$. A similar theorem holds for $\theta = \pi$.*

Proof. Let us assume there is an asymptotic angle θ_r contained in (2.3) which is equivalent to $\theta = 0$. Two possibilities exist either θ_r is independent of R and hence $b_{s,r} = 0$ or else θ_r depends on R and there exists at least one $b_{s,r} \neq 0$.

For the first possibility $\theta_r = r\pi/m$ and

$$S(R, 0) - S(R, \theta_r) = \sum_{k=1}^{m} a_k\, R^k\, (1 - \cos k\theta_r). \tag{2.6}$$

Hence

$$\lim_{R\to\infty} [S(R, 0) - S(R, \theta_r)]$$

is finite if and only if

$$\cos k\theta_r = 1 \tag{2.7}$$

for all values of k, $1 \leqslant k \leqslant m$, for which $a_k \neq 0$. Since the greatest common divisor of the above-mentioned values of k is one it is easily shown that the only solution of equations (2.7) in the range $0 \leqslant \theta_r \leqslant \pi$ is $\theta_r = 0$.

Let us now assume that the constants $b_{s,r}$ are not all zero. This implies that there must exist a value of k in (2.2), for which $a_k \neq 0$, and such that $\sin k(r\pi/m) \neq 0$. Otherwise $\theta_r = r\pi/m$ would be a solution of (2.2) and we would be back to the case where θ_r is independent of R. Let us suppose that $k = q$ is the largest value of k such that

$$\sin(qr\pi/m) \neq 0, \quad a_q \neq 0, \qquad 1 \leqslant q \leqslant m - 1. \tag{2.8}$$

Under these conditions we can easily show from (2.2) that

$$\theta_r = \frac{r\pi}{m} + O(1/R^{m-q}). \tag{2.9}$$

We shall now consider values of $k > q$ for which $a_k \neq 0$. For all such values of k we have by assumption that

$$\sin(kr\pi/m) = 0. \tag{2.10}$$

Hence by using (2.8), (2.9) and expanding $\cos(k\theta_r)$ we have

$$\cos(k\theta_r) = \cos(kr\pi/m) + O(1/R^{2m-2q}). \tag{2.11}$$

This implies that

$$a_k R^k \cos k\theta_r = a_k R^k \cos(kr\pi/m) + O(R^{k+2q-2m}). \tag{2.12}$$

Since $k \leqslant m$ and $q \leqslant m - 1$, $k + 2q - 2m \leqslant q - 1$. Hence the order term cannot affect any terms of order R^s, $s \geqslant q$. A similar result can be obtained for the single term $k = q$. This means that

$$\sum_{k=1}^{m} a_k R^k \cos k\theta_r = \sum_{k=q}^{m} a_k R^k \cos(kr\pi/m) + O(R^{q-1}). \tag{2.13}$$

From (2.13) we can conclude that

$$\lim_{R\to\infty} [S(R, 0) - S(R, \theta_r)]$$

is finite only if

$$\cos(kr\pi/m) = 1 \tag{2.14}$$

for all values of k, $q \leqslant k \leqslant m$, such that $a_k \neq 0$. However $a_q \neq 0$ by assumption. Hence (2.14) implies $\cos(qr\pi/m) = 1$. This in turn means $\sin(qr\pi/m) = 0$ which contradicts (2.8). Hence if $\theta = 0$ is an asymptotic angle of $S(R, \theta)$ it has no equivalent asymptotic angles. A similar proof can be given for $\theta = \pi$.

THEOREM 2.2. *Let θ_i and θ_j be two non-equivalent asymptotic angles of $S(R, \theta)$. Then*

$$S(R, \theta_i) - S(R, \theta_j) = O(R^q), \tag{2.15}$$

where q is some integer in the range $1 \leqslant q \leqslant m - 1$.

Proof. Using (2.3) one can easily see that $S(R, \theta_i)$, $S(R, \theta_j)$ have expansions in the powers of R of the form

$$S(R, \theta_i) = \sum_{-\infty}^{m} C_s R^s, \tag{2.16}$$

$$S(R, \theta_j) = \sum_{-\infty}^{m} \bar{C}_s R^s, \tag{2.17}$$

where C_s and $\bar{C}_s$ are constants such that $C_m = \bar{C}_m$. Since the maxima are not equivalent there must be a value of s for which $C_s \neq \bar{C}_s$, $1 \leqslant s \leqslant m - 1$. If we denote by q the largest such value of s then

$$S(R, \theta_i) - S(R, \theta_j) = O(R^q). \tag{2.18}$$

Definition 2.4. Let $\phi_1 < \phi_2 < \ldots < \phi_t$ be the subset of asymptotic angles which includes the asymptotic angle for which $S(R, \theta)$ is equal to its greatest maximum and all asymptotic angles which are equivalent to this latter asymptotic angle. The asymptotic angles ϕ_i $(i = 1, 2, \ldots, t)$ are called the *maximal asymptotic angles* of $S(R, \theta)$.

THEOREM 2.3. *The set of maximal asymptotic angles consists only of one angle if either $\theta = 0$ or $\theta = \pi$ is a maximal asymptotic angle.*

This follows immediately from Theorem 2.1.

THEOREM 2.4. *Let us assume that the set of maximal angles consists of a single angle ϕ_1, such that $0 < \phi_1 < \pi$. If ϵ is defined by*

$$\epsilon = R^{(1-4m)/3}, \tag{2.19}$$

then

$$S(R, \theta) \leqslant S(R, \phi_1 - \epsilon) \tag{2.20}$$

for all values of θ in the range $0 \leqslant \theta \leqslant \phi_1 - \epsilon$ or $\phi_1 + \epsilon \leqslant \theta \leqslant \pi$.

The proofs for the two ranges of θ are similar so that we shall consider only the range $0 \leqslant \theta \leqslant \phi_1 - \epsilon$. In this range we note that there must always exist at least one minimum of $S(R, \theta)$. This is easily seen because $\theta = 0$ is always a solution of (2.2). Hence $S(R, 0)$ is either a maximum or minimum of $S(R, \theta)$. If $S(R, 0)$ is a minimum then there is nothing more to prove. If $S(R, 0)$ is a maximum then there is a minimum in the range $0 \leqslant \theta \leqslant \phi_1$ because $S(R, \phi_1)$ is also by assumption a maximum. However all the maxima and minima are determined by (2.3), hence the angles at which they occur are always a finite distance apart. Since $\epsilon \to 0$ as $R \to \infty$ we must have that there always exists at least one minimum of $S(R, \theta)$ in the range $0 \leqslant \theta \leqslant \phi_1 - \epsilon$.

If there is only one minimum of $S(R, \theta)$ in the range $0 \leqslant \theta \leqslant \phi_1 - \epsilon$ and if it occurs at $\theta = 0$ then the proof of the theorem is trivial. In this case $S(R, \theta)$ is an increasing function in the range $0 \leqslant \theta \leqslant \phi_1 - \epsilon$ and hence for this range $S(R, \theta) \leqslant S(R, \phi_1 - \epsilon)$.

Let us now assume that there are other minima of $S(R, \theta)$ in this range and denote by $\psi > 0$ the angle closest to ϕ_1 for which $S(R, \theta)$ is a minimum. Since $S(R, \theta)$ is increasing in the range $\psi \leqslant \theta \leqslant \phi_1 - \epsilon$ we must have $S(R, \theta) \leqslant S(R, \phi_1 - \epsilon)$ in this range. Hence to complete the proof we need only consider the range $0 \leqslant \theta \leqslant \psi$. Again it is easily shown that in this later range a maximum of $S(R, \theta)$ must exist. We denote by α the angle at which the greatest maximum of $S(R, \theta)$ occurs for the range $0 \leqslant \theta \leqslant \psi$. We note of course that α is an asymptotic angle. Clearly for the range $0 \leqslant \theta \leqslant \psi$, $S(R, \theta) \leqslant S(R, \alpha)$. Hence

$$S(R, \phi_1 - \epsilon) - S(R, \theta) \geqslant S(R, \phi_1 - \epsilon) - S(R, \alpha). \tag{2.21}$$

Expanding $S(R, \phi_1 - \epsilon)$ in a Taylor's expansion, and remembering that $S'(R, \phi_1) = 0$ because $S(R, \phi_1)$ is a maximum, we must have

$$S(R, \phi_1 - \epsilon) = S(R, \phi_1) + \tfrac{1}{2}S''(R, \phi_1)\,\epsilon^2 + \ldots. \tag{2.22}$$

Since $S(R, \phi_1)$ is a maximum $S''(R, \phi_1)$ is negative and is of order $O(R^m)$ in R. Hence using (2.19)

$$S(R, \phi_1 - \epsilon) = S(R, \phi_1) - O(R^{1/4}). \tag{2.23}$$

From (2.21) and (2.23)

$$S(R, \phi_1 - \epsilon) - S(R, \theta) \geqslant S(R, \phi_1) - S(R, \alpha) - O(R^{1/4}). \tag{2.24}$$

However, ϕ_1, α are non-equivalent asymptotic angles, hence, by Theorem 2.2,

$$S(R, \phi_1) - S(R, \alpha) = O(R^q). \tag{2.25}$$

where q is an integer in the range $1 \leqslant q \leqslant m - 1$. Thus (2.24) and (2.25) imply

$$S(R, \phi_1 - \epsilon) - S(R, \theta) \geqslant O(R^q). \tag{2.26}$$

By assumption $S(R, \phi_1)$ is the greatest maximum of $S(R, \theta)$ in the range $0 \leqslant \theta \leqslant \pi$. Hence the order term must be positive. When this is so (2.26) gives

$$S(R, \theta) \leqslant S(R, \phi_1 - \epsilon). \tag{2.27}$$

This completes the proof for the range $0 \leqslant \theta \leqslant \phi_1 - \epsilon$. The proof for the range $\phi_1 + \epsilon \leqslant \theta \leqslant \pi$ is similarly obtained.

THEOREM 2.5. *Let us assume that the set of maximal asymptotic angles* $\phi_1 < \phi_2 < \ldots < \phi_t$ *consists of at least two angles. Further let us assume that* ϕ_i *and* ϕ_{i+1} *are two consecutive maximal asymptotic angles. If* ϵ *is given by* (2.19) *then one of the following inequalities must hold*:

(2.28) $$S(R, \theta) \leqslant S(R, \phi_i + \epsilon)$$

or

(2.29) $$S(R, \theta) \leqslant S(R, \phi_{i+1} - \epsilon)$$

for all values of θ in the range $\phi_i + \epsilon \leqslant \theta \leqslant \phi_{i+1} - \epsilon$.

We omit the proof of this theorem because of its similarity to the proof used in Theorem 2.4.

THEOREM 2.6. *Let us assume that the conditions of Theorem* 2.4 *are satisfied. Further let* I_1, I_2 *be defined by*

(2.30) $$I_1 = \int_0^{\phi_1 - \epsilon} \exp[P(Re^{i\theta}) - P(Re^{i\phi_1}) - in\theta]\, d\theta,$$

(2.31) $$I_2 = \int_{\phi_1 + \epsilon}^{\pi} \exp[P(Re^{i\theta}) - P(Re^{i\phi_1}) - in\theta]\, d\theta,$$

where n *is any real number. Then a number* $k > 0$ *exists such that*

(2.32) $$|I_1| \leqslant \exp(-kR^{1/4})$$

(2.33) $$|I_2| \leqslant \exp(-kR^{1/4}).$$

Proof.

$$|I_1| \leqslant \int_0^{\phi_1 - \epsilon} \exp[S(R, \theta) - S(R, \phi_1)]\, d\theta.$$

By Theorem (2.4), $S(R, \theta) \leqslant S(R, \phi_1 - \epsilon)$. Hence

(2.28) $$|I_1| \leqslant \pi \exp\{S(R, \phi_1 - \epsilon) - S(R, \phi_1)\}.$$

However, using (2.23) we have

(2.29) $$|I_1| \leqslant \pi \exp(-kR^{1/4}). \qquad k > 0.$$

We may absorb the π into the exponent and write

(2.30) $$|I_1| \leqslant \exp(-kR^{1/4}). \qquad k > 0.$$

The proof for I_2 is similar.

THEOREM 2.7. *Let us assume that the conditions of Theorem* (2.5) *are satisfied. Under these conditions the absolute value of each of the integrals*:

(2.31) $$I_0 = \int_0^{\phi_1 - \epsilon} \exp[P(Re^{i\theta}) - P(Re^{i\phi_j}) - in\theta]\, d\theta,$$

(2.32) $$I_i = \int_{\phi_i + \epsilon}^{\phi_{i+1} - \epsilon} \exp[P(Re^{i\theta}) - P(Re^{i\phi_j}) - in\theta]\, d\theta, \quad i = 1, 2, \ldots, t-1,$$

(2.33) $$I_t = \int_{\phi_t + \epsilon}^{\pi} \exp[P(Re^{i\theta}) - P(Re^{i\phi_j}) - in\theta]\, d\theta,$$

is at most of order $\exp\{-o(R^{1/4})\}$ *no matter which maximal asymptotic angle* ϕ_j *is chosen.*

The proof is essentially the same as that used in Theorem 2.6.

THEOREM 2.8. *Let $P(x)$ be given by* (1.4) *and as usual $S(R, \theta)$ denotes the dominant function of $P(x)$, given by* (2.1). *We shall denote by τ the complex number*

$$\tau = Re^{i\theta} \tag{2.34}$$

and by Θ the operator

$$\Theta = \tau \frac{d}{d\tau}. \tag{2.35}$$

If n is a sufficiently large real number then the equation

$$\Theta P(\tau) = n \tag{2.36}$$

has a unique solution for τ corresponding to each asymptotic angle of $S(R, \theta)$. Further the solution $\tau_r(n)$ corresponding to the asymptotic angle θ_r, given by (2.3), satisfies the equations

$$\arg \tau_r = \theta_r \tag{2.37}$$

and

$$|\tau_r(n)| = R_r(n) \sim (n/m|a_m|)^{1/m}. \tag{2.38}$$

Proof. From (1.4)

$$\Theta P(\tau) = \sum_{k=1}^{m} k a_k \tau^k = \sum_{k=1}^{\bar{m}} k a_k R^k \cos(k\theta) - i S'(R, \theta). \tag{2.39}$$

Hence equating the real and imaginary parts of (2.36) we find

$$\sum_{k=1}^{\bar{m}} k a_k R^k \cos k\theta = n \tag{2.40}$$

and

$$S'(R, \theta) = 0. \tag{2.41}$$

Equation (2.41) is of course the same as (2.2) hence all of the solutions of (2.41) for θ are given by (2.3). Let us choose a particular θ_r which is also an asymptotic angle of $S(R, \theta)$. We have already seen that this implies $a_m \cos r\pi = |a_m|$. If we substitute (2.3) into (2.40) and expand into powers of R we see that (2.40) can be written

$$m|a_m| R^m + O(R^{m-1}) = n. \tag{2.42}$$

Although the first term is independent of r the order term will in general be dependent on r. From (2.42) it is easily seen that for large values of n the solution of (2.42) for a real positive value of R is unique for each fixed value of r. Denoting this solution by $R_r(n)$ it is also easily seen

$$R_r(n) \sim (n/m|a_m|)^{1/m}. \tag{2.43}$$

Since $\theta_r = \arg \tau_r(n)$ and $R_r(n) = |\tau_r(n)|$ the theorem is proven. One may also show without too much difficulty that if $R_r(n)$ and $R_j(n)$ are two different solutions for $|\tau|$ corresponding to different asymptotic angles that

$$\lim_{n \to \infty} [R_r(n) - R_j(n)] = 0.$$

In concluding this section we might mention that only the maximal asymptotic angles are necessary in developing our asymptotic formula for B_n.

3. An example of equivalent maxima. By Theorem 2.3 we know that if $\theta = 0$ is a maximal asymptotic angle that $S(R, \theta)$ has no other maximal asymptotic angles. This result is, of course, only true when we assume that the greatest common divisor of the values of k, for which $a_k \neq 0$, is one. For a long time the authors conjectured that under the above assumption any dominant function $S(R, \theta)$ would have only one maximal asymptotic angle. Since the conjecture proved to be false it is of interest to give an example in which the dominant function has two such angles.

Let us take $P(x)$ to be given by

$$P(x) = x^{90} + x^{30} - x^{24} + x^{15} - x^{10}. \tag{3.1}$$

Clearly the G.C.D. (90, 30, 24, 15, 10) = 1. The dominant of (3.1) is given by

$$S(R, \theta) = R^{90}\cos(90\theta) + R^{30}\cos(30\theta) - R^{24}\cos(24\theta) + R^{15}\cos(15\theta) - R^{10}\cos(10\theta). \tag{3.2}$$

(3.3) Clearly the solutions of $S'(R, \theta) = 0$ which correspond to maxima of $S(R, \theta)$ are of the form

$$\theta_r = \frac{r\pi}{45} + O(1/R^{60}), \qquad r = 0, 1, 2, \ldots, 45. \tag{3.4}$$

Hence

$$S(R, \theta_r) = S(R, r\pi/45) + O(1/R^{30}). \tag{3.5}$$

This means that the maximal asymptotic angles can be obtained by choosing those values of r which will make $S(R, r\pi/45)$ as large as possible. In each case the coefficient of R^{90} is one. All the other coefficients of powers of R depend on r. In order to make $S(R, r\pi/45)$ as large as possible we start with coefficient of R^{30} and make this coefficient as large as possible. Then, in turn, we deal with each of the coefficients of the lower powers.

$$S(R, r\pi/45) = R^{90} + R^{30}\cos(2r\pi/3) - R^{24}\cos(8r\pi/15) + R^{15}\cos(r\pi/3) - R^{10}\cos(2r\pi/9) \tag{3.6}$$

Step A. $\cos(2r\pi/3)$. The values of r in the set 0, 1, 2, . . . , 45 which make $\cos(2r\pi/3) = 1$ are $r = 0, 3, 6, \ , 45$.

Step B. $-\cos(8r\pi/15)$. The values of r in the set 0, 3, 6, . . . , 45 which make $-\cos(8r\pi/15)$ as large as possible are 6, 9, 21, 24, 36, 39.

For these values $-\cos(8r\pi/15) = \cos(\pi/5)$.

Step C. $\cos(r\pi/3)$. The values of r of the set 6, 9, 21, 24, 36, 39 which make $\cos(r\pi/3) = 1$ are $r = 6, 24, 36$.

Step D. $-\cos(2r\pi/9)$. The values of r of the set 6, 24, 36 which make $-\cos(2r\pi/9)$ as large as possible are $r = 6, 24$.

Hence there will be two maximal asymptotic angles corresponding to the values $r = 6$ and $r = 24$.

It is interesting to note that if the sign of x^{10} in $P(x)$ is changed to positive then we would again have had a unique maximal asymptotic angle. Namely, the one corresponding to $r = 36$.

4. Asymptotic formula for unique maximal asymptotic angles. When $\theta = 0$ is a maximal asymptotic angle of the dominant function we have seen that there are no other maximal asymptotic angles. For this case the derivation of an asymptotic formula for B_n, as given by (1.3), can be obtained by using the same procedure as was used in our previous paper **(1)**. The proofs and final formulae are identical. For this reason we state, without proof, that the first term of the asymptotic formula is

$$B_n \sim \frac{n!\exp(P(R))}{R^n}(2\pi A^2 P(R))^{-\frac{1}{2}}, \tag{4.1}$$

where A is the operator

$$A = R\frac{d}{dR} \tag{4.2}$$

and R as a function of n is given by

$$A\,P(R) = n. \tag{4.3}$$

Other terms of the asymptotic formula can be obtained from the general formula given in **(1)**.

Further the fact that

$$A^2 P(R) = \sum_{k=1}^{m} k^2 a_k R^k \tag{4.4}$$

allows us to reduce (4.1) to

$$B_n \sim \frac{n!\exp(P(R))}{(2\pi)^{\frac{1}{2}} m(a_m)^{\frac{1}{2}} R^{n+\frac{1}{2}m}}, \tag{4.5}$$

with R again given by (4.3).

The case $\theta = \pi$ can be reduced to the case $\theta = 0$ by replacing θ by $\pi - \theta$. This is equivalent to replacing R by $-R$. For this case the first term of the asymptotic formula for B_n is given by

$$B_n \sim \frac{(-1)^n \exp(P(-R))}{R^n}(2\pi A^2 P(-R))^{-\frac{1}{2}} \tag{4.6}$$

and R as a function of n is given by

$$A\,P(-R) = n. \tag{4.7}$$

The formula that corresponds to (4.5) is

$$B_n \sim \frac{(-1)^n \exp(P(-R))}{(2\pi|a_m|)^{\frac{1}{2}} mR^{n+\frac{1}{2}m}}, \tag{4.8}$$

and R is given by (4.7).

We now proceed to discuss the case when the dominant function has a unique maximal asymptotic angle say at

$$\theta = a(R), \quad a(R) \neq 0, \quad a(R) \neq \pi. \tag{4.9}$$

Throughout we shall simply write

$$a(R) = a. \tag{4.10}$$

From (1.3) we have

$$B_n = \frac{d^n}{dx^n} (\exp(P(x)))\Big]_{x=0}. \tag{4.11}$$

By Cauchy's Theorem

$$B_n = \frac{n!}{2\pi i} \int_C z^{-(n+1)} \exp(P(z))\, dz, \tag{4.12}$$

where C is chosen as the circle $z = Re^{i\theta}$. At this stage the radius of the circle R is an arbitrary positive number. Equation (4.12) can be written

$$\begin{aligned} B_n &= \frac{n!}{2\pi R^n} \int_{-\pi}^{\pi} \exp[P(Re^{i\theta}) - in\theta]\, d\theta \\ &= \frac{n!}{\pi R^n} \Re \int_0^{\pi} \exp[P(Re^{i\theta}) - in\theta]\, d\theta. \end{aligned} \tag{4.13}$$

If we define the complex number τ by

$$\tau = Re^{ia} \tag{4.14}$$

then (4.13) can be put into the form

$$B_n = \frac{n!}{\pi} \Re\left[\tau^{-n} \exp(P(\tau)) \int_0^{\pi} \exp[P(\tau \exp(i(\theta - a))) - P(\tau) - in(\theta - a)]\, d\theta \right]. \tag{4.15}$$

We shall show that the integral of (4.15) has an asymptotic expansion in terms of powers of $1/\tau$. However by using Theorem (2.6) we can easily show that the integrals

$$\int_0^{a-\epsilon} \exp[P(\tau \exp(i(\theta - a))) - P(\tau) - in(\theta - a)]\, d\theta \tag{4.16}$$

and

$$\int_{a+\epsilon}^{\pi} \exp[P(\tau \exp(i(\theta - a))) - P(\tau) - in(\theta - a)]\, d\theta \tag{4.17}$$

are both of order $\exp(-O(|\tau|^{1/4}))$ when ϵ is given by (2.19). Anticipating this result we may write

$$B_n \sim \frac{n!}{\pi} \Re\left[\tau^{-n} \exp P(\tau) \int_{a-\epsilon}^{a+\epsilon} \exp\{P(\tau \exp i(\theta - a)) - P(\tau) - in(\theta - a)\}\, d\theta \right]. \tag{4.18}$$

We define I to be the integral contained in (4.18) and replace $\theta - a$ by θ.

Hence

$$I = \int_{-\epsilon}^{\epsilon} \exp(F(\theta))\, d\theta, \tag{4.19}$$

where

$$F(\theta) = P(\tau e^{i\theta}) - P(\tau) - in\theta. \tag{4.20}$$

Remembering that Θ is the operator given by

$$\Theta = \tau d/d\tau \tag{4.21}$$

we have on expanding $F(\theta)$ about $\theta = 0$ that

$$F(\theta) = (\Theta P(\tau) - n)\, i\theta - \Theta^2 P(\tau) \frac{\theta^2}{2} + \sum_{k=3}^{\infty} \Theta^k P(\tau) \frac{(i\theta)^k}{k!}. \tag{4.22}$$

We have assumed that there exists a unique maximal asymptotic angle $\theta = a$ and by Theorem (2.8) there corresponds a unique solution of the equations

$$\Theta P(\tau) = n \tag{4.23}$$

$$\arg \tau = a. \tag{4.24}$$

By choosing τ to be this unique solution we may write (4.22) as

$$F(\theta) = -\Theta^2 P(\tau) \frac{\theta^2}{2} + \sum_{k=3}^{\infty} \Theta^k P(\tau) \frac{(i\theta)^k}{k!}. \tag{4.25}$$

Since $0 < \arg \tau < \pi$ there is no loss in generality in considering the complex plane of τ as cut along the negative real axis. For this reason there is no ambiguity in finding the square root of τ. Similarly we shall find that there is no ambiguity in finding the square root of certain polynomials in τ which later enter into the discussion.

For simplicity we introduce the following notation:

$$\tau = Z^{-2}, Z = \tau^{-\frac{1}{2}}; \tag{4.26}$$

$$\phi = \theta(\tfrac{1}{2}\Theta^2 P(\tau))^{\frac{1}{2}}; \tag{4.27}$$

$$h = \epsilon(\tfrac{1}{2}\Theta^2 P(\tau))^{\frac{1}{2}}; \tag{4.28}$$

$$C_k(Z) = \sum_{s=1}^{m} s^k a_s Z^{2m-2s}; \tag{4.29}$$

$$f_k(Z) = C_k(Z)\, Z^{m(k-2)}\, (\tfrac{1}{2}C_2(Z))^{-k/2}; \tag{4.30}$$

$$\psi(Z, \phi) = \sum_{k=3}^{\infty} f_k(Z) \frac{(i\phi)^{\frac{1}{2}k}}{k!}. \tag{4.31}$$

In this notation the substitution (4.27) reduces the integral (4.19) to the form

$$I = \left(\frac{2}{\Theta^2 P(\tau)}\right)^{\frac{1}{2}} \int_{-h}^{h} \exp[-\phi^2 + \psi(Z, \phi)]\, d\phi. \tag{4.32}$$

We note that although θ is a real variable that ϕ as given by (4.27) is, in general, complex. Similarly h is, in general, a complex variable. From Theorem (2.8) we know that

$$\lim_{n\to\infty} |\tau| = \lim_{n\to\infty} R = \infty. \tag{4.33}$$

It is easily shown that $|h| = O(|\tau|^{1/8})$ and hence

$$\lim_{n\to\infty} |h| = \infty. \tag{4.34}$$

Finally one can show without too much difficulty that

$$\lim_{n\to\infty} \arg h = 0. \tag{4.35}$$

Our next step is to expand $\exp(\psi(Z, \phi))$ into a Maclaurin expansion about $Z = 0$, of the form

$$\exp(\psi(Z, \phi)) = \sum_{k=0}^{\infty} \Psi_k(\phi)\, Z^k, \qquad \Psi_0 = 1, \tag{4.36}$$

where the $\Psi_k(\phi)$ are polynomials in ϕ. Quite formally one would have on integrating term by term and replacing h by ∞ that

$$I \sim \left(\frac{2}{\Theta^2 P(\tau)}\right)^{\frac{1}{2}} \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} e^{-\phi^2} \Psi_k(\phi)\, d\phi\, Z^k. \tag{4.37}$$

The rigorous justification of (4.37) requires only a slight modification of the procedure that was used in our previous paper **(1)**. For this reason we omit this justification. Defining b_k to be given by

$$b_k = \int_{-\infty}^{\infty} e^{-\phi^2} \Psi_k(\phi)\, d\phi, \tag{4.38}$$

we can obtain from (4.18) and (4.37) that

$$B_n \sim \pi^{-1} 2^{\frac{1}{2}} n! \, \Re\left[\tau^{-n} (\Theta^2 P(\tau))^{-\frac{1}{2}} \exp(P(\tau)) \sum_{k=0}^{\infty} b_k \tau^{-\frac{1}{2}k}\right] \tag{4.39}$$

and τ is given as the solution of

$$\Theta P(\tau) = n, \tag{4.40}$$

which corresponds to the unique (by assumption) maximal asymptotic angle of the dominant function of $P(x)$.

Since $b_0 = (\pi)^{\frac{1}{2}}$ the first term of (4.39) is

$$B_n \sim 2^{\frac{1}{2}}(n!)\, \Re[\tau^{-n} (\pi\, \Theta^2 P(\tau))^{-\frac{1}{2}} \exp(P(\tau))]. \tag{4.41}$$

We can, if we so desire, replace $(\Theta^2(P(\tau)))^{\frac{1}{2}}$ by

$$(\Theta^2 P(\tau))^{\frac{1}{2}} \sim m |a_m|^{\frac{1}{2}} |\tau|^{\frac{1}{2}m}; \tag{4.42}$$

hence

$$B_n \sim n! \, \Re[\tau^{-n} (2\pi^{-1})^{\frac{1}{2}} \exp(P(\tau))/m|a_m|^{\frac{1}{2}} |\tau|^{\frac{1}{2}m}]. \tag{4.42}$$

In concluding this section we would like to point out that formula (4.39) does not hold for $a = 0$ or $a = \pi$. The reason being that a maximal asymptotic angle internal to the range $0 < \theta < \pi$ gives a contribution to the asymptotic formula from both sides of the maximal asymptotic angle. However at the boundary points of the interval a contribution to the asymptotic formula is obtained from one side only. For this reason when $a = 0$ or π formula (4.39) and the subsequent formulas are out by a factor of $\frac{1}{2}$.

5. The general asymptotic formula. In the general case we have seen that the dominant function can have more than one maximal asymptotic angle. We assume that there is at least two such angles and denote the set of maximal asymptotic angles by ϕ_j $(j = 1, 2, \ldots, l;\ l \geq 2)$. Under these assumptions neither of the angles 0 or π can be a member of the set.

From Theorem (2.8) we have seen that corresponding to a fixed value of j there corresponds a unique solution for τ of the equations

$$\Theta P(\tau) = n, \tag{5.1}$$

$$\arg \tau = \phi_j. \tag{5.2}$$

Let us denote this solution by

$$\tau_j = R_j e^{i\phi_j}. \tag{5.3}$$

Without difficulty one can show that R_j, R_k are two different solutions for $|\tau|$ such that

$$\lim_{n=\infty} (R_j - R_k) = 0. \tag{5.4}$$

Now B_n is given by

$$B_n = \frac{n!}{2\pi i} \int_C z^{-(n+1)} \exp(P(z))\, dz, \tag{5.5}$$

where C is any closed contour enclosing the origin. Instead of a circular contour we choose C to be given as follows:

A. *Range* $0 \leq \theta \leq \phi_1 + \epsilon_1$, where $\epsilon_1 = R_1^{(1-4m)/8}$. In this range C is the circular arc $z = R_1 e^{i\theta}$.

B. *Range*

$$\phi_j + \epsilon_j \leq \theta \leq \phi_{j+1} + \epsilon_{j+1},\ 1 \leq j \leq l-1, \quad \epsilon_j = R_j^{(1-4m)/8}.$$

For this range C is the circular arc $z = R_{j+1} e^{i\theta}$.

C. *Range*

$$\phi_l + \epsilon_l \leq \theta \leq \pi, \quad \epsilon_l = R_l^{(1-4m)/8}.$$

Here C is the circular arc $z = R_l e^{i\theta}$.

D. In order to make a closed contour we join all the circular arcs by radial lines at the end points of each arc.

E. *Range* $-\pi \leq \theta \leq 0$. The contour C for this range is taken to be the mirror image of the contour for $0 \leq \theta \leq \pi$.

By using Theorems (2.5) and (2.7) we can show that only the portions of the contour $\phi_j - \epsilon_j \leqslant \theta \leqslant \phi_j + \epsilon_j$, $j = 1, 2, \ldots, t$ and their mirror images contribute to the asymptotic formula. Hence

$$B_n \sim \frac{n!}{\pi} \Re\left[\sum_{j=1}^{t} \int_{\phi_j - \epsilon_j}^{\phi_j + \epsilon_j} \frac{\exp[P(R_j e^{i\theta}) - in\theta]\, d\theta}{R_j^n} \right]. \tag{5.6}$$

Each of the maximal asymptotic angles can then be treated individually by the method of the previous section. This leads to the asymptotic formula

$$B_n \sim \frac{n!}{\pi} \Re\left[\sum_{k=0}^{\infty} \sum_{j=1}^{t} b_{kj}\, \tau_j^{-n} \exp(P(\tau_j))\, \tau_j^{-\frac{1}{2}k}\, (\tfrac{1}{2}\Theta^2 P(\tau_j))^{-\frac{1}{2}k} \right], \tag{5.7}$$

where $\Theta P(\tau_j)$ means $[\Theta P(\tau)]_{\tau=\tau_j}$. Similarly for $\Theta^2 P(\tau_j)$. Further the τ_j are given as the solutions of the equations

$$\Theta P(\tau) = n \tag{5.8}$$

and

$$\arg \tau = \phi_j, \tag{5.9}$$

where ϕ_j is a maximal asymptotic angle of the dominant function of $P(x)$. Finally the b_{kj} will have formulae analogous to (4.38). To obtain explicit formulae, expansions about each maximal asymptotic angle are involved.

6. Example. We have chosen as an example to illustrate our method the Hermite polynomials $H_n(t)$. Szegö **(3**; p. 194**)** gives an asymptotic expansion of these polynomials and the method of proof divides the expansion formula into two cases according as n is even or odd. Our method makes no such separation and the two cases are treated as one.

The generating function of $H_n(t)$ is

$$\exp(2tx - x^2) = \sum_{n=0}^{\infty} H_n(t) \frac{x^n}{n!}. \tag{6.1}$$

In the notation of the present paper:

$$P(x) = 2tx - x^2, \tag{6.2}$$

$$S(R, \theta) = 2tR \cos\theta - R^2 \cos 2\theta. \tag{6.3}$$

It is easily seen that the dominant function has a single maximum at $\theta = \phi_1$ where ϕ_1 is given by

$$\phi_1 = \arccos(t/2R). \tag{6.4}$$

The value of τ is determined by

$$2t\tau - 2\tau^2 = n, \tag{6.5}$$

$$\arg \tau = \phi_1. \tag{6.6}$$

Hence

(6.7) $$R = (n/2)^{\frac{1}{2}},$$

(6.8) $$\phi_1 = \arccos\,(t/(2n)^{\frac{1}{2}}),$$

(6.9) $$\tau = Re^{i\phi_1} = \tfrac{1}{2}(t + i(2n - t^2)^{\frac{1}{2}}).$$

From (6.8) for all real t and large n

(6.10) $$\phi_1 = \tfrac{1}{2}\pi - t\,(2n)^{-\frac{1}{2}} - (t^3(2n)^{-3/2}/6)$$

and

(6.11) $$n\,\phi_1 = \tfrac{1}{2}\,n\pi - (\tfrac{1}{2}n)^{\frac{1}{2}}t + \text{terms of order } n^{-\frac{1}{2}}.$$

We shall drop terms of the order $n^{-\frac{1}{2}}$ in our final formula to get an expression for the first term in the asymptotic expansion. At certain stages one must retain such terms. For example to obtain (6.11) from (6.10).

From (6.7), (6.9) and (6.11)

(6.12) $$\tau^n = (\tfrac{1}{2}n)^{\frac{1}{2}n}\,e^{i((\frac{1}{2}n\pi - (\frac{1}{2}n)^{\frac{1}{2}}t + \ldots)}.$$

Similarly

(6.13) $$\exp(P(\tau)) = \exp[\tfrac{1}{2}\,(t^2 + n) + i((\tfrac{1}{2}n)^{\frac{1}{2}}t \ldots)],$$

(6.14) $$[\Theta^2\,P(\tau)]^{\frac{1}{2}} = (2n)^{\frac{1}{2}} + \ldots.$$

Hence by (4.41)

(6.15) $$H_n(t) \sim (2/\pi)^{\frac{1}{2}}\,n!\,\Re[(\tfrac{1}{2}n)^{-\frac{1}{2}n}\,(\exp\,\tfrac{1}{2}(t^2 + n) + i((2n)^{\frac{1}{2}}\,t - \tfrac{1}{2}n\pi + \ldots)](2n)^{-\frac{1}{4}}.$$

Using Stirling's formula for $n!$

(6.16) $$H_n(t) \sim 2^{\frac{1}{4}(n+1)}(n/e)^{\frac{1}{2}n}\,e^{\frac{1}{4}t^2}\cos((2n)^{\frac{1}{2}}\,t - \tfrac{1}{4}n\pi).$$

This term agrees to the proper order with Szegö's formula. Other terms can easily be calculated and it can be seen that our method does not distinguish between even and odd n.

We hope in subsequent papers to apply the method to other problems involving asymptotic expansions.

7. Conclusion. In concluding this paper we would like to point out several possibilities for further generalizations. The most obvious generalization would be to obtain an asymptotic formula when the coefficients a_k of $P(x)$ are allowed to be complex functions of a complex parameter t. We have carried this problem far enough to be reasonably certain that our method will generalize to this case with a minimum of modification. A second type of generalization would be to allow the a_k to be functions of n and obtain formulas of the Plancherel-Rotach type. We have found examples of this type of problem for which our method will apply but it does not seem likely that the method can be used to solve the general problem of this type. A final type of

generalization we have considered is the replacement of the polynomial $P(x)$ by a function of $f(x)$. We have found that the method can be applied to the special case of the Bell numbers **(2)**, in which $f(x) = e^x - 1$, but fairly stringent conditions would have to be placed on $f(x)$ in order for our method to apply to an arbitrary class of functions.

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A SIMPLE ALGORITHM FOR FINDING MAXIMAL NETWORK FLOWS AND AN APPLICATION TO THE HITCHCOCK PROBLEM

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Introduction. The network-flow problem, originally posed by T. Harris of the Rand Corporation, has been discussed from various viewpoints in **(1; 2; 7; 16)**. The problem arises naturally in the study of transportation networks; it may be stated in the following way. One is given a network of directed arcs and nodes with two distinguished nodes, called *source* and *sink*, respectively.[1] All other nodes are called *intermediate*. Each directed arc in the network has associated with it a nonnegative integer, its *flow capacity*. Source arcs may be assumed to be directed away from the source, sink arcs into the sink. Subject to the conditions that the flow in an arc is in the direction of the arc and does not exceed its capacity, and that the total flow into any intermediate node is equal to the flow out of it, it is desired to find a maximal flow from source to sink in the network, i.e., a flow which maximizes the sum of the flows in source (or sink) arcs.

Thus, if we let P_1 be the source, P_n the sink, we are required to find x_{ij} $(i, j = 1, \ldots, n)$ which maximize

$$\sum_{i=2}^{n} x_{1i} \tag{1}$$

subject to

$$\sum_{j} (x_{ij} - x_{ji}) = 0 \qquad (i = 2, \ldots, n-1), 0 \leqslant x_{ij} \leqslant c_{ij}, \tag{2}$$

where the c_{ij} are given nonnegative integers.

This is, of course, a linear programming problem, and hence may be solved by Dantzig's simplex algorithm **(3)**. In fact, the simplex computation for a problem of this kind is particularly efficient, since it can be shown that the sets of equations one solves in the process are always triangular **(1)**. However, for the flow problem, we shall describe what appears to be a considerably more efficient algorithm.

Of some theoretical interest is the fact that the procedure assures one of obtaining a strict increase in the total flow at each step (in contrast with the simplex method). In addition, the Hitchcock transportation problem **(4; 12; 14)** can be solved via the flow algorithm in a way which naturally generalizes

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[1] A problem in which there are several sources and sinks, with flows permitted from any source to any sink, is reducible to a single-source, single-sink problem.

the combinatorial method recently proposed by Kuhn **(15)** for the optimal-assignment problem. Informal tests by hand indicate that this way of solving the Hitchcock problem is extremely efficient.

1. The minimal cut theorem. A nonconstructive proof of the minimal cut theorem, which asserts equality of maximal flow value and minimal cut value,[2] has been given by the present writers in **(7)**. Subsequently a constructive proof based on the simplex criterion of linear programming was developed **(1)**. The algorithm which we describe in the next section also provides a constructive proof of the theorem in case the c_{ij} are integral (or rational),[3] as we have assumed for this paper. Like the simplex algorithm, it not only produces a maximal flow but a minimal cut as well. This will be important for our application to the Hitchcock problem.

It should perhaps be pointed out that an *undirected problem*, by which we mean that the directions of flow in intermediate arcs are not specified, so that the capacity constraints on these arcs are of the form

$$x_{ij} + x_{ji} \leqslant c_{ij} \qquad (i, j = 2, \ldots, n-1;\ i < j), \tag{3}$$

presents nothing new, since we may replace each undirected arc by a pair of oppositely directed arcs, each with capacity equal to that of the original arc; i.e., replace (3) by

$$x_{ij} \leqslant c_{ij}, \quad x_{ji} \leqslant c_{ij}. \tag{4}$$

For, given $X' = (x_{ij}')$ satisfying (4) and the conservation equations at intermediate nodes, setting

$$x_{ij} = \max(x_{ij}' - x_{ji}', 0) \tag{5}$$

yields an equivalent flow $X = (x_{ij})$ satisfying these equations and (3).

The minimal cut theorem is true for undirected networks as well as for directed networks, or more generally, for mixed networks in which some intermediate arcs are directed, others not, the obvious changes in definitions of cuts and chains having been made. This follows from the comments above and the fact, easily proved, that the minimal cut value is the same for a mixed network and its equivalent directed network. The theorem is still valid when capacity constraints on nodes are admitted, where a cut now, of course, includes nodes as well as arcs. This may be proved by splitting each node into two nodes as suggested in Fig. 1, and placing the capacity c of the old node on the new directed arc joining the two new nodes, thus obtaining an equivalent network with capacity constraints on arcs only. For a direct proof of the theorem in this general form, see **(1)**. It is also shown there (and follows

[2] A *cut* is a collection of arcs which meets every directed chain from source to sink; the *value* of a cut is the sum of capacities of its members. This definition of a cut corresponds to that of *disconnecting set* given in **(7)**.

[3] No restriction of this kind is made in **(1, 7)**.

Fig. 1

from the algorithm of the next section) that for integral c_{ij}, an integral maximizing flow exists.

There are several well-known combinatorial theorems which can be viewed as rather direct consequences of the minimal cut theorem and the existence of an integral maximizing flow. Among these we mention Menger's theorem on linear graphs **(13)**, Dilworth's chain decomposition theorem for partially ordered sets **(5)**, and Hall's theorem concerning set representatives **(11)**. For proofs, see **(1; 8; 9)**.

2. The algorithm. We start the computation from any convenient integral flow whatsoever. The initial flow is used to define a starting matrix $A = (a_{ij})$ by letting a_{ij} be the capacity of the arc from P_i to P_j diminished by the flow from P_i to P_j and increased by the flow from P_j to P_i. If no other flow is readily available, one may start with the zero flow, corresponding to $A = C$, where c_{ij} is the original capacity from P_i to P_j.

For certain values of $j = 1, \ldots, n$ we shall define labels v_j, μ_j recursively as follows.

Let $v_1 = \infty$, $\mu_1 = 0$. For those j such that $a_{1j} > 0$, define $v_j = a_{1j}$, $\mu_j = 1$. In general, from those i which have received labels v_i, μ_i but which have not previously been examined, select an i and scan for all j such that $a_{ij} > 0$ and v_j, μ_j have not been defined. For these j, define

$$v_j = \min(v_i, a_{ij}), \quad \mu_j = i. \tag{6}$$

Continue this process until v_n, μ_n have been defined, or until no further definitions may be made and v_n, μ_n have not been defined. In the latter case the computation ends. In the former case, proceed to obtain a new a_{ij} matrix as follows.

Replace $a_{\mu_n n}$ by $a_{\mu_n n} - v_n$ and $a_{n\mu_n}$ by $a_{n\mu_n} + v_n$. In general, replace $a_{\mu_j j}$ by $a_{\mu_j j} - v_n$ and $a_{j\mu_j}$ by $a_{j\mu_j} + v_n$, where each j is the $\mu_{j'}$ of the preceding j' in the backward replacement. This replacement continues until $\mu_j = 1$ has been completed. The labels v_j, μ_j are then recomputed on the basis of the new A matrix and the process is repeated.

Notice that v_n is a positive integer, and hence the process terminates. Upon termination, the maximal flow X is given by defining

$$x_{ij} = \max(c_{ij} - a_{ij}, 0), \tag{7}$$

as we now prove.

LEMMA 1. *X is a flow.*

Proof. Clearly $0 \leqslant x_{ij} \leqslant c_{ij}$, since $a_{ij} \geqslant 0$. It remains to show that X satisfies $\sum_j (x_{ij} - x_{ji}) = 0$ $(i = 2, \ldots, n-1)$. Now the process ensures that $c_{ij} + c_{ji} = a_{ij} + a_{ji}$. It follows from this and the definition of X that

$$\sum_j (x_{ij} - x_{ji}) = \sum_j (c_{ij} - a_{ij}).$$

It therefore suffices to prove that $\sum_j a_{ij}$ $(i = 2, \ldots, n-1)$ is invariant under the computation, as certainly $\sum_j (c_{ij} - a_{ij}) = 0$ for the starting matrix A obtained from a flow X. But if i $(\neq 1, n)$ is the μ_l of some l, then there is a $k = \mu_i$. Thus, for this i, the new a_{ij}' are either equal to the old a_{ij} or are given by

$$a_{ij}' = \begin{cases} a_{ij} - v_n & \text{for } j = l, \\ a_{ij} + v_n & \text{for } j = k, \\ a_{ij} & \text{otherwise;} \end{cases}$$

hence $\sum_j a_{ij}' = \sum_j a_{ij}$.

LEMMA 2. *X is a maximal flow.*

Proof. At the point where termination occurs, we have defined a set S of nodes, consisting of those nodes P_i for which v_i, μ_i have been defined; and further, $P_1 \in S$, $P_n \notin S$. Consider the set Γ of directed arcs P_iP_j such[4] that $P_i \in S$, $P_j \notin S$. Clearly $a_{ij} = 0$ for such pairs i, j, as otherwise we would have defined v_j, μ_j.

We will show that Γ is a cut whose value is equal to $\sum_j x_{1j}$, thus proving that X is a maximal flow and Γ a minimal cut.

That Γ is a cut is clear. For if there were a chain

$$P_1P_{i_1} \ldots P_{i_k}P_n, \text{ with the arcs } P_1P_{i_1} \notin \Gamma, \ldots, P_{i_k}P_n \notin \Gamma,$$

then we could deduce successively (since $P_1 \in S$) that

$$P_{i_1} \in S, \ldots, P_n \in S,$$

which is a contradiction. To see that Γ is equal to the flow value, notice that

$$\sum_j (c_{ij} - a_{ij}) = \begin{cases} 0 & (1 < i < n), \\ \sum_j x_{1j} & (i = 1). \end{cases}$$

Now sum these equations over those i for which $P_i \in S$. On the left side, if P_i and P_j are both in S, then $c_{ij} - a_{ij}$ and $c_{ji} - a_{ji}$ are both in the summation and are negatives of each other. All that remain are terms of the form $c_{ij} - a_{ij}$, where $P_i \in S$, $P_j \notin S$. For these, as we pointed out above, $a_{ij} = 0$. Thus the sum on the left reduces precisely to the sum of capacities of members of Γ, as was to be shown.

[4] The set Γ is actually the set of "left arcs" defined in (7).

3. The Hitchcock problem. The Hitchcock transportation problem arises frequently in applications of linear programming. We shall propose another computation for the problem which will amount to solving a sequence of flow problems of a particularly simple kind. The basic idea, which stems from a proof given by Egerváry **(6)** for a theorem of König **(13**, p. 232**)** on linear graphs, has been used by Kuhn **(15)** to develop a very efficient combinatorial algorithm for the optimal assignment problem, a special case of the Hitchcock problem. Our method differs only in details from the Kuhn algorithm in this case.

We take the Hitchcock problem in the following form: Given a matrix $D = (d_{ij})$ of nonnegative integers, and two sets of nonnegative integers $(a_1, \ldots, a_m)$, $(b_1, \ldots, b_n)$, with

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j,$$

it is desired to find a matrix $X = (x_{ij})$ satisfying the constraints

$$x_{ij} \geqslant 0, \quad \sum_j x_{ij} = a_i, \quad \sum_i x_{ij} = b_j \tag{8}$$

which minimizes the linear form

$$\sum_{i,j} d_{ij} x_{ij}. \tag{9}$$

A physical interpretation of the problem is that there are m originating points for a commodity, the ith point having a_i units, and n destinations, the jth one requiring b_j units. If d_{ij} is the cost, per unit of commodity, of shipping from origin i to destination j, find a shipping programme of minimal cost.

The dual (see **(10)** for linear programming duality theorems) of the Hitchcock problem is: Find α_i, β_j satisfying the constraints

$$\alpha_i + \beta_j \leqslant d_{ij} \qquad (i = 1, \ldots, m; \quad j = 1, \ldots, n) \tag{10}$$

which maximize

$$\sum_i a_i \alpha_i + \sum_j b_j \beta_j. \tag{11}$$

The proof that the algorithm to be described yields a solution to the Hitchcock problem will be based on the fact that the dual form (11) increases by at least one unit with the solution of each successive flow problem. Since (11) is bounded above by (9), this suffices.

Each of the flow problems will be of the following form. Find $X = (x_{ij})$ satisfying

$$\begin{aligned} &x_{ij} \geqslant 0, \ \sum_j x_{ij} \leqslant a_i, \ \sum_i x_{ij} \leqslant b_j, \\ &x_{ij} = 0 \text{ for a given set } \Omega \text{ of pairs } i, j, \end{aligned} \tag{12}$$

which maximizes

$$\sum_{i,j} x_{ij}. \tag{13}$$

This is actually a special Hitchcock problem. To see that it is also a flow problem, set up the directed network of $m + n + 2$ nodes shown in Fig. 2, where the capacity on the directed arc P_iQ_j is zero if $i, j \in \Omega$, large otherwise, and the capacities of source and sink arcs are the a_i and b_j, as shown, and interpret x_{ij} as the flow from P_i to Q_j.

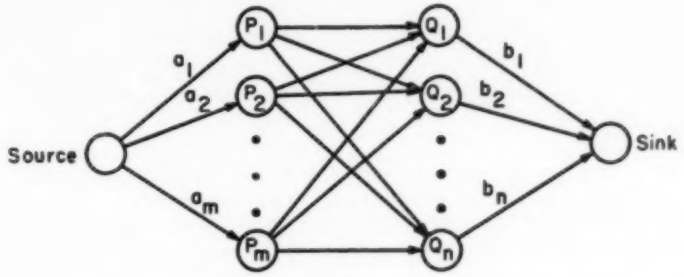

Fig. 2

To solve such a problem we may of course use the computation of §2 involving, in this case, an $m + n + 2$ by $m + n + 2$ matrix. It is possible (and computationally convenient) to describe the process in terms of an m by n array. The verification that the two descriptions are the same for this particular class of problems will be left to the reader.

Let X be an integral solution of the constraints (12). Corresponding to certain of the rows $i = 1, \ldots, m$ of X we will define integers v_i, μ_i; similarly, for certain of the columns $j = 1, \ldots, n$ we will define integers w_j, λ_j. These definitions will be made recursively; first a set of v_i, μ_i will be defined, from these a set of w_j, λ_j will be defined, and so forth, alternating between the rows and columns.

For those i such that $\sum_j x_{ij} < a_i$, define

$$v_i = a_i - \sum_j x_{ij}, \mu_i = 0. \tag{14}$$

Next select an i which has been labelled and scan for all (unlabelled) j such that $i, j \notin \Omega$; for these j, define

$$w_j = v_i, \lambda_j = i. \tag{15}$$

Repeat until the previously labelled i's are exhausted. We then select a labelled j and scan for unlabelled i such that $x_{ij} > 0$; for these i, define

$$v_i = \min(x_{ij}, w_j), \ \mu_i = j. \tag{16}$$

Repeat until the previously labelled j's are exhausted. Again select one of the i's just labelled, look for unlabelled j such that $i, j \notin \Omega$, and define w_j, λ_j by (15). Continue in this fashion, using (15) and (16) alternately until either

w_j, λ_j have been defined for some j with $\sum_i x_{ij} < b_j$, or until no further definitions may be made. In the latter case X is maximal by Lemma 2; in the former we can get an improvement as follows. Let

$$v = \min(w_j, b_j - \sum_i x_{ij}). \tag{17}$$

Alternately add and subtract v from the sequence

$$x_{\lambda_j j}, x_{\lambda_j j_1}, x_{i_1 j_1}, x_{i_1 j_2}, \ldots, x_{i_{k-1} j_k}, x_{i_k j_k}, \tag{18}$$

where

$$j_1 = \mu_{\lambda_j}, i_1 = \lambda_{j_1}, j_2 = \mu_{i_1}, i_2 = \lambda_{j_2}, \ldots, j_k = \mu_{i_{k-1}}, i_k = \lambda_{j_k},$$

and $\mu_k = 0$.

i \ j	1	2	3	4	5	6	7	8	9	10	a_i	v_i	μ_i
1			(1)	(1)							2	1	3
2							(2)		(1)		3		
3		(1)								◯	1	1	2
4	(1)				(3)						4	1	1
5		◯			(3)		◯	◯		◯	3		
6		◯	◯					(2)			3	1	0
7	◯	◯							(1)		1	1	9
8		◯		(1)		◯					1	1	4
9								◯	◯	(3)	5	2	0
10				(6)		(2)					8	1	4
b_j	1	1	1	8	9	2	2	2	2	3			
w_j	1	1	1	1	1	1		2	2	2			
λ_j	7	6	6	1	4	10		9	9	9			

Fig. 3

Example. In Fig. 3, cells containing a circle comprise $\bar{\Omega}$ (the complement of Ω). The entries within the circles (zeros elsewhere) constitute an X satisfying the constraints (12). The defining process terminates with $w_5 = 1$, $\lambda_5 = 4$, since column 5 is a column in which the sum of the entries x_{i5} is less than b_5; i.e., $3 + 3 < 9$. The sequence along which an improvement of

$$\min(w_5, b_5 - \sum_i x_{i5}) = 1$$

can be made is $x_{45} = 3$, $x_{41} = 1$, $x_{71} = 0$, $x_{79} = 1$, $x_{99} = 0$, as is easily read off using the λ's and μ's alternately.

We are now in a position to describe a general routine for the Hitchcock problem. As a starting point, form the difference matrix

(19) $$d_{ij} - \alpha_i - \beta_j,$$

where

$$\alpha_i = \min_j d_{ij}, \beta_j = \min_i (d_{ij} - \alpha_i).$$

Thus $d_{ij} - \alpha_i - \beta_j \geqslant 0$; i.e., α_i, β_j satisfy the dual constraints (10). The next step is to solve the flow problem with

(20) $$\Omega = \{ij | d_{ij} - \alpha_i - \beta_j > 0\}.$$

If the maximizing flow X satisfies

$$\sum_{i,j} x_{ij} = \sum_i a_i$$

then X is a minimizing solution to the original Hitchcock problem.

This is easily deduced as follows. Observe first of all that for any α_i, β_j, the two Hitchcock problems with cost matrices d_{ij} and $d_{ij} - \alpha_i - \beta_j$ are equivalent; for $\sum_j x_{ij} = a_i$, $\sum_i x_{ij} = b_j$ imply that

$$\sum_{i,j} (d_{ij} - \alpha_i - \beta_j) x_{ij} = \sum_{i,j} d_{ij} x_{ij} - \sum_i a_i \alpha_i - \sum_j b_j \beta_j,$$

and the last two sums on the right are independent of x_{ij}. Thus if we have α_i, β_j with $d_{ij} - \alpha_i - \beta_j \geqslant 0$, and are able to find an X satisfying (8) and $x_{ij} = 0$ for $ij \in \Omega$, then clearly X minimizes $\sum_{i,j} (d_{ij} - \alpha_i - \beta_j) x_{ij}$, hence minimizes $\sum_{i,j} d_{ij} x_{ij}$.

If, on the other hand, $\sum_{i,j} x_{ij} < \sum_i a_i$, let I be the index set of the labelled rows of X, J the index set of the labelled columns, and define new dual variables by

(21) $$\alpha_i' = \begin{cases} \alpha_i + k & (i \in I), \\ \alpha_i & (i \notin I), \end{cases}$$
$$\beta_j' = \begin{cases} \beta_j - k & (j \in J), \\ \beta_j & (j \notin J), \end{cases}$$

where $k = \min(d_{ij} - \alpha_i - \beta_j)$, $i \in I$, $j \notin J$. Notice that $k > 0$, since pairs i, j with $i \in I$, $j \notin J$ are contained in Ω.

LEMMA 3. $\sum_i a_i \alpha_i' + \sum_j b_j \beta_j' > \sum_i a_i \alpha_i + \sum_j b_j \beta_j$.

Proof. The fact that X is a maximal flow implies that $\sum_i x_{ij} = b_j$ for $j \in J$. Also, the labelling process ensures that $x_{ij} = 0$ for $i \notin I$, $j \in J$, and, as we have mentioned earlier, $x_{ij} = 0$ for $i \in I$, $j \notin J$. Since all i with $\sum_j x_{ij} < a_i$ are in I, it follows that

$$\sum_{i \in I} a_i > \sum_{i \in I} \sum_{j \in J} x_{ij} = \sum_{j \in J} b_j,$$

and hence the new dual form has been increased by the amount

$$k\left(\sum_{i \in I} a_i - \sum_{j \in J} b_j\right) > 0.$$

Another way to see this is to note that the minimal cut in the associated network (Fig. 2) has value $\sum_{i \in I} a_i + \sum_{j \in J} b_j < \sum_{i} a_i$ and hence

$$\sum_{i \in I} a_i > \sum_{j \in J} b_j.$$

Now form the new difference matrix $d_{ij} - \alpha_i - \beta_j \geqslant 0$ by subtracting k from the I-rows of the previous difference matrix, and adding k to the J-columns. We may now take the maximal flow X of the previous flow problem as a starting point in the new flow problem and proceed as before.

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ON A DIOPHANTINE PROBLEM

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Introduction. If $a_1, a_2, \ldots, a_k$ are relatively prime positive integers then the equation

$$a_1x_1 + \ldots + a_kx_k = n \tag{1}$$

always has solutions in non-negative x_i for n sufficiently large. Sylvester called the number of non-negative solutions of (1) the *denumerant* of the equation. We shall denote the denumerant by $n(a_1, \ldots, a_k)$.

We define $N_j(a_1, \ldots, a_k)$ to be the smallest positive integer such that $n(a_1, \ldots, a_k) > j$ for all $n > N_j(a_1, \ldots, a_k)$.

Using the known result

$$n(1, 2, 3) = [(n^2 + 6n + 15)/12]$$

we can readily show that

$$N_j(1, 2, 3) = [\sqrt{(12j - 6)} - 2].$$

In general the computation of $N_j(a_1, \ldots, a_k)$ is a difficult job. Our interest here is in $N_1(a_1, \ldots, a_k)$. The present author has given **(4)** the value of $N_1(a_1, \ldots, a_k)$ when the a_i are in arithmetic progression. Brauer and Seelbinder **(2; 3)** have given various upper bounds for $N_1(a_1, \ldots, a_k)$ in the general case. In this paper we concern ourselves with an upper bound for $N_1(m, m + a, m + b)$ where $(a, b) = 1$. Our upper bound is in many cases best possible. We also give a theorem concerning an upper bound for

$$N_1(m, m + a_1, \ldots, m + a_k)$$

where $a_1, a_2, \ldots, a_k$ are relatively prime positive integers.

1. Three lemmas. Throughout this section

$$0 < a < b,\ (a, b) = 1,\ P = (a - 1)(b - 1).$$

Since $N_1(a, b) = P$ **(1**, p. 124**)**, the equation $ax + by = n$ is solvable in non-negative integers for all $n > P$. The non-negative solution with smallest x is denoted by x_n, y_n. If x, y is any non-negative solution of $ax + by = n$ then

$$x + y \geqslant x_n + y_n.$$

Also

$$x_{n+b} + y_{n+b} = x_n + y_n + 1.$$

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Since $(a, b) = 1$, the numbers αa for $0 \leqslant \alpha \leqslant b - 1$ constitute a complete system of residues modulo b. Hence $n \equiv \alpha a \pmod{b}$ has a unique solution with $0 \leqslant \alpha \leqslant b - 1$. Denote this solution by α_n.

LEMMA 1. $x_n = \alpha_n, \quad y_n = (n - a\alpha_n)/b.$

Proof. Clearly

$$a\alpha_n + b((n - a\alpha_n)/b) = n.$$

Hence all solutions of $ax + by = n$ are given by

$$\alpha_n + bt, \ (n - a\alpha_n)/b - at.$$

For non-negative solutions we must have

$$t \geqslant -\alpha_n/b > -1.$$

Hence all non-negative solutions have $x \geqslant \alpha_n$.

LEMMA 2. *If* $n > P$, $\alpha_n = b - 1$ *and* $1 \leqslant c \leqslant n - P$ *then*

$$x_n + y_n \geqslant x_{n-c} + y_{n-c} + 1.$$

Proof. We have

$$\begin{aligned} x_n + y_n &= \alpha_n + (n - a\alpha_n)/b = (n + \alpha_n(b - a))/b \\ &= (n + (b - 1)(b - a))/b > (n - c + \alpha_{n-c}(b - a))/b \\ &= x_{n-c} + y_{n-c}. \end{aligned}$$

LEMMA 3.

$$\max_{n \in K} (x_n + y_n) \leqslant b - 2 + [m/b]$$

where K *consists of those* n *satisfying* $P \leqslant n \leqslant P + m - 1$, $m \geqslant 1$.

Proof. Let B be the smallest integer greater than $P + m - 1$ which has $\alpha_B = b - 1$. Then

$$B > P + m - 1 \geqslant n$$

and therefore $B - n \geqslant 1$. Also $B - n \leqslant B - P$. Hence taking the n, c of Lemma 2 to be B, $B - n$ respectively, we see that

$$x_B + y_B \geqslant x_{B-(B-n)} + y_{B-(B-n)} + 1 = x_n + y_n + 1.$$

This is true for all $n \in K$, so

$$\max_{n \in K} (x_n + y_n) \leqslant x_B + y_B - 1 = (B + \alpha_B(b - a))/b - 1.$$

For arbitrary m we have

$$B = P + [m/b]\, b - 1 + b.$$

Hence

$$\begin{aligned} \max_{n \in K} (x_n + y_n) &\leqslant (P + [m/b]\, b - 1 + b + (b - 1)(b - a))/b - 1 \\ &= b - 2 + [m/b]. \end{aligned}$$

2. The main theorem.

THEOREM 1. $N_1(m, m+a, m+b) \leqslant m(b-2+[m/b])+(a-1)(b-1)$ *where* $0 < a < b$, $(a, b) = 1$, $m \geqslant 2$.

Proof. Let P, K, x_n, y_n be as in §1. Define

$$Q = \max_{n \in K} (x_n + y_n).$$

Now define $\bar{x}_j$, $\bar{y}_j$, $\bar{z}_j$ for $j \geqslant P$ by the following:

$$\bar{x}_j = x_i, \ \bar{y}_j = y_i \text{ for } j \equiv i \pmod{m}, \qquad P \leqslant i \leqslant P+m-1,$$
$$\bar{z}_j = Q + [(j-P)/m] - \bar{x}_j - \bar{y}_j.$$

Then

$$m\bar{z}_j + (m+a)\bar{x}_j + (m+b)\bar{y}_j = m(\bar{z}_j + \bar{x}_j + \bar{y}_j) + a\bar{x}_j + b\bar{y}_j$$
$$= mQ + m[(j-P)/m] + a\bar{x}_j + b\bar{y}_j. \tag{2}$$

As j runs over

$$P + sm, \ P + sm + 1, \ldots, P + sm + m - 1$$

for $s \geqslant 0$, the right side of this equation runs over

$$mQ + ms + P, \ mQ + ms + P + 1, \ldots, mQ + ms + P + m - 1.$$

Hence as j runs over the integers greater than or equal to P and s runs over the integers greater than or equal to 0, the left side of (2) runs over the integers greater than or equal to $P + mQ$. Replacing Q by its upper bound from Lemma 3 now gives the desired result.

Essentially the same proof yields the following:

THEOREM 2. $N_1(m, m+a_1, \ldots, m+a_k) \leqslant P + mQ$ *where* $a_1, a_2, \ldots, a_k$ *are relatively prime positive integers,*

$$P = N_1(a_1, \ldots, a_k), \quad Q = \max_{n \in K} (x_{1n} + \ldots + x_{kn}),$$

K *is the set of* n *such that* $P \leqslant n \leqslant P + m - 1$, *and* $x_{1n}, \ldots, x_{kn}$ *is a non-negative solution of* (1) *with smallest sum.*

3. Special cases. Let $\bar{N}_1(m, m+a, m+b)$ denote the upper bound in Theorem 1. We compare $\bar{N}_1$ with N_1 in a few special cases.

(a) By the main result in **(4**; see also **2**, Theorem 7, p. 310**)**,

$$N_1(m, m+1, m+2) = m[\tfrac{1}{2}m] = \bar{N}_1(m, m+1, m+2).$$

(b) By a result stated in **(4)**,

$$N_1(m, m+1, m+b) = \bar{N}_1(m, m+1, m+b)$$

for

$$m \equiv -1 \pmod{b}, \quad m \geqslant b^2 - 5b + 3;$$

and

$$N_1(m, m+1, m+b) = \bar{N}_1(m, m+1, m+b) - (m - b[m/b])$$

for

$$m \equiv -1 \pmod{b}, \quad m > b^2 - 4b + 2.$$

(c) By direct evaluation it is not difficult to show

$$N_1(m, m+2, m+3) = \bar{N}_1(m, m+2, m+3).$$

(d) By computation for $2 < m < 16$ we find

$$N_1(m, m+2, m+5) = \bar{N}_1(m, m+2, m+5)$$

for

$$m = 5, 8, 10, 13, 14, 15$$

and not for

$$m = 2, 3, 4, 6, 7, 9, 11, 12, 16.$$

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SETS AND SUBSERIES

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Suppose $\alpha_n \neq 0$ for all n. By replacing a set of terms from the series $\sum\alpha_n$ by 0's, we form a subseries of the original series. These subseries can be put in one to one correspondence with the non-terminating binary expansion of the points on the real line segment (0, 1). The point $\xi = .b_1b_2 \ldots b_n \ldots$ shall correspond to a subseries if and only if $b_n = 0$ whenever the nth term of the original series has been replaced by 0 and $b_n = 1$ whenever the nth term is retained. We can now speak of sets of subseries of the first category, measure zero, etc.

If we have a series $\sum a_n$ whose terms $\{a_n\}$ form a positive monotone decreasing sequence, $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n \geqslant \ldots$, then it is easy to show by a Dedekind section the existence of an index p such that $\sum a_n^q$ converges for $q > p$ and diverges for $q < p$. The object of this note is to show that if the terms of a series are a positive monotone decreasing sequence, the set of subseries that have an index differing from the original series is of the first category.

We first prove a lemma.

LEMMA. *If $\{c_n\}$ is a positive monotone decreasing sequence, $\sum c_n$ diverges,* $\limsup nc_n \neq 0$, *then the set of subseries that converges is of the first category.*

Proof. We can assume that $2n(m)c_{2n(m)} > k$ where $\{n(m)\}$ is a subsequence of $\{n\}$. However, this implies $n(m)c_{n(m)} > \frac{1}{2}k$ and $\nu c_\nu > \frac{1}{2}k$ for $n(m) \leqslant \nu \leqslant 2\,n(m)$. We can associate with this sequence a regular summation method A, defined by

$$t_m = \frac{1}{n(m)} \sum_{\nu=n(m)}^{2\,n(m)} s_\nu .$$

Suppose now that $c'_n = c_n$ if the term is retained for the subseries and $c'_n = 0$ otherwise. For any subseries $\sum c'_n$ we shall choose a sequence of 0's and 1's, $\{s'_n\}$, so that $s'_n = 1$ if $c'_n \neq 0$ and $s'_n = 0$ otherwise. There is an evident correspondence between the sequence $\{s'_n\}$ and the point corresponding to the subseries. We now see that if $\sum c'_n$ converges, $\{s'_n\}$ must be A summable to 0. For if $t_{m_\mu} \geqslant \lambda > 0$ for an infinite set $\{\mu\}$, then

$$\sum_{\nu=n(m_\mu)}^{2\,n(m_\mu)} c'_\nu > \lambda\, n(m_\mu)\, c_{2n(m_\mu)} > \lambda \frac{k}{2}$$

and the series diverges.

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However, it has been shown by Hill that the set of points on (0, 1) in the binary expansion corresponding to such a set of 0's and 1's is of the first category.

We are now ready to prove our theorem.

THEOREM. *If $\{a_n\}$ is a positive monotone decreasing sequence, the set of subseries of $\sum a_n$ with a different index is of the first category.*

Proof. The index for the series $\sum a_n$ may be assumed to be 1. For if the index is p, $\sum a_n^p$ will have an index 1 and the set with a different index will be the same for $\sum a_n$ and $\sum a_n^p$.

If $r < 1$, then $\limsup n\, a_n^r > 0$, for if

$$\lim_{n \to \infty} n\, a_n^r = 0,$$

then $a_n = 0(n^{-1/r})$ and for $r < q < 1$, $\sum a_n^q$ would converge, contrary to our assumption. Hence the set of subseries of $\sum a_n^r$ that converges for any r is of the first category.

If a subseries has an index less than 1, then it will belong to the set E_ν of convergent subseries of

$$\sum a_n^{1-1/\nu}, \qquad \nu = 2, 3, \ldots$$

for some ν. The set of all subseries with index less than 1 will be contained in the union of these sets and so will be of the first category.

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SHORT PROOF OF A MAP-COLOUR THEOREM

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Heawood **(3)** showed that for $h > 1$ the countries of any map on a surface of connectivity $h = 3 - \chi$ can be coloured using at most

$$n_h = [\tfrac{1}{2}(7 + \sqrt{(24h - 23)})]$$

colours. In a previous paper **(1)** I proved the following

THEOREM. *For $h = 3$ and for $h \geqslant 5$ a map on a surface of connectivity h with chromatic number n_h always contains n_h mutually adjacent countries.*

A new short proof of this will now be given. It is based on the following result **(2)**:

If a critical k-chromatic graph contains $N > k$ nodes and E edges then

$$2E \geqslant (k - 1)N + k - 3. \tag{1}$$

(A graph is called critical if deleting any arbitrary node or edge reduces the chromatic number. A k-chromatic graph always contains a critical k-chromatic subgraph, and a critical graph is finite and connected **(1**, p. 481**)**.)

To prove the above map-colour theorem it is sufficient to show that for $h = 3$ and $h \geqslant 5$, if a critical k-chromatic graph with more than k nodes is drawn without intersection of edges on a surface of connectivity h, then $k \leqslant n_h - 1$. Let such a graph have N nodes and E edges. By (1) and Euler's Theorem **(1**, 2.1**)**, we have

$$(k - 1)N + k - 3 \leqslant 2E \leqslant 6N + 6(h - 3). \tag{2}$$

Further **(1**, 1.3**)**, $N \geqslant k + 2$.

For the case when $h = 3$, we observe that $n_3 = 7$ and, from (2), $k \leqslant 6$. For $h \geqslant 5$ it may be assumed that $k > 6$, so it follows from (2) that

$$(k - 7)(k + 2) \leqslant (k - 7)\,N \leqslant 6h - 15 - k,$$

whence

$$k \leqslant 2 + [\sqrt{(6h + 3)}].$$

Thus $k \leqslant n_h - 1$ when

$$[\sqrt{(6h + 3)}] \leqslant [\tfrac{1}{2}(1 + \sqrt{(24h - 23)})].$$

This is clearly true if

$$\sqrt{(6h + 3)} \leqslant \tfrac{1}{2}(1 + \sqrt{(24h - 23)}),$$

that is, if $h \geqslant 13$. It can be verified also for $h = 5, 6, \ldots, 12$. This completes the proof of the Theorem.

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CORRECTION TO "A MINIMUM-MAXIMUM PROBLEM FOR DIFFERENTIAL EXPRESSIONS"

D. S. CARTER

The author takes this opportunity to correct some misprints and to add a note to his paper "*A minimum-maximum problem for differential expressions*," in this Journal, 9 (1957), 132-140.

Page 134, Equation (2.5): for this equation read

$$\| \mathfrak{L}x_0 \| = \inf \{ \| \mathfrak{L}x \| \mid x \epsilon X \}.$$

Page 137, line −5: for "e_0^4" read "e^4".

Page 138, line −7: for "$|\xi_0^4|$" read "$|\zeta_0|$".

Added note: Since the preparation of this manuscript it has come to the author's attention that the present problem bears a close relationship to the "Bang-Bang" control problem **(3)**. Choosing $c = 0$, $\eta_b = 0$, and allowing the endpoint b to vary, it is easy to show that the value of $||g_0||$ at the solution is a continuous monotone function of b. The value of b for which $||g_0|| = 1$ provides the solution to a "Bang-Bang" problem of a rather general type.

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UPPER AND LOWER BOUNDS FOR THE AREA OF A TRIANGLE FOR WHOSE SIDES TWO SYMMETRIC FUNCTIONS ARE KNOWN

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1. Introduction. Improving on inequalities given by Gerretsen **(2)**, Beatty **(1)** has proved that for the area Δ of any plane triangle with sides a, b, c the following inequalities hold:

$$\frac{(K-H)^2}{12} > \Delta^2 > \frac{(K-H)(3K-5H)}{12} \tag{1.1}$$

where

$$H = \tfrac{1}{2}(a^2+b^2+c^2), \quad K = bc+ca+ab; \tag{1.2}$$

the signs of equality in (1.1) only apply when the triangle is equilateral. Beatty has also remarked that the second inequality in (1.1) is of no value in case $5H > 3K$, since then the lower estimate which it gives for Δ^2 is not even positive.

As an improvement on the foregoing estimates, a proof of the following inequalities will be given here (§2):

$$\frac{s(s-q)^2(s+2q)}{27} > \Delta^2 > \frac{s(s+q)^2(s-2q)}{27}, \tag{1.3}$$

where

$$q = (a^2+b^2+c^2-bc-ca-ab)^{\frac{1}{2}}, \tag{1.4}$$

i.e.

$$q = (\tfrac{1}{2}Q)^{\frac{1}{2}}, \tag{1.5}$$

where

$$Q = (b-c)^2+(c-a)^2+(a-b)^2 \tag{1.6}$$

is the measure of "unequilaterality" already considered by Gerretsen **(2)**; and

$$s = \frac{a+b+c}{2} = \left(\frac{H+K}{2}\right)^{\frac{1}{2}} \tag{1.7}$$

is the semiperimeter of the triangle.[1]

The first (second) equality sign in (1.3) holds for isosceles triangles whose base is the smallest (largest) of the three sides; of course both equality signs apply when the triangle is equilateral, since then $q = 0$.

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[1]The use of s and q as the given symmetric functions of the sides throughout the present note (instead of Beatty's H and K) seems justified by the fact that it simplifies the formulation and proof of (1.3).

In our (1.3), as in Beatty's (1.1), the second inequality is useless when $5H \geq 3K$, i.e. when $q \geq \frac{1}{2}s$. It is easily seen that in this case triangles can be found with as small an area as one might wish; hence no other lower bound for Δ can then exist than the trivial one: $\Delta > 0$.

Apart from this exceptional case (and that of an equilateral triangle), (1.3) represents an improvement over (1.1), which might be rewritten in terms of s and q as follows:

$$\frac{(s^2-q^2)^2}{27} > \Delta^2 > \frac{(s^2-q^2)(s^2-4q^2)}{27}, \tag{1.8}$$

by using the easily verified formulae

$$H = \tfrac{1}{3}(2s^2+q^2), \quad K = \tfrac{1}{3}(4s^2-q^2). \tag{1.9}$$

Indeed, Beatty's upper bound in (1.8) is higher than ours in (1.3), since the difference between them is

$$\frac{(s^2-q^2)^2}{27} - \frac{s(s-q)^2(s+2q)}{27} = \frac{q^2(s-q)^2}{27} > 0; \tag{1.10}$$

analogously from

$$\frac{s(s+q)^2(s-2q)}{27} - \frac{(s^2-q^2)(s^2-4q^2)}{27} = \frac{2q^2(s+q)(s-2q)}{27} > 0 \tag{1.11}$$

($q < \frac{1}{2}s$) it follows that our lower bound in (1.3) is higher than Beatty's in (1.8).

Incidentally it might be remarked that a new simple proof of (1.1) is thus obtained by deriving (1.8) from (1.3).

Moreover, it can be shown that in a certain sense our estimates (1.3) are best possible; indeed, the bounds given by (1.3) for Δ^2 are nothing else than the extremal values that can be attained by Δ^2 considered as a function of the three positive variables a, b, c subject to the constraining relations (1.4) and (1.7) and furthermore to the obvious inequalities

$$b+c > a, \quad c+a > b, \quad a+b > c, \tag{1.12}$$

equivalent to

$$a < s, \quad b < s, \quad c < s. \tag{1.13}$$

Of course, the given values of s and q (besides being positive) have to satisfy the inequality

$$q < s \tag{1.14}$$

(equivalent to Beatty's $H < K$) as a consequence of the easily verified identity

$$q^2 = s^2 - \frac{3}{s}\left(abc + \frac{\Delta^2}{s}\right). \tag{1.15}$$

To prove the foregoing statements it would only be necessary to find these extremal values of Δ^2 by standard methods of differential calculus, although some caution is necessary because of the inequalities (1.13).[2] It seems, however, that the proof of (1.3) given in §2 avoids this difficulty, and at the same time has the advantage of being purely algebraic. (The use of trigonometric functions is not essential, as will be pointed out at the end of §2.)

A final remark (§3) will be devoted to the problem of finding bounds for the product of the three sides of a triangle.

2. Proof of the inequalities (1.3). We start with the well known formulae **(3**, pp. 25–28**)** for the trigonometric solution of a reduced cubic equation

$$x^3 + 3Cx + D = 0 \tag{2.1}$$

with real roots x_1, x_2, x_3 satisfying the condition

$$x_1 + x_2 + x_3 = 0, \tag{2.2}$$

viz. (if the roots are numbered conveniently):

$$\begin{cases} x_1 = 2\sqrt{-C}\cos\Theta, \; x_2 = 2\sqrt{-C}\cos(\Theta + 120°), \\ \qquad\qquad x_3 = 2\sqrt{-C}\cos(\Theta + 240°), \end{cases} \tag{2.3}$$

where the auxiliary angle Θ is defined by

$$\cos 3\Theta = -\tfrac{1}{2}D(-C)^{-3/2}, \quad 0 \leqslant \Theta \leqslant 60°. \tag{2.4}$$

A possible choice fulfilling (2.2) is

$$x_1 = a - \tfrac{2}{3}s, \quad x_2 = b - \tfrac{2}{3}s, \quad x_3 = c - \tfrac{2}{3}s; \tag{2.5}$$

then

$$3C = x_2x_3 + x_3x_1 + x_1x_2 \tag{2.6}$$

will be related to q—defined in (1.4)—by

$$C = -q^2/9, \tag{2.7}$$

since

$$q^2 = \frac{(b-c)^2 + (c-a)^2 + (a-b)^2}{2} = \frac{(x_2 - x_3)^2 + (x_3 - x_1)^2 + (x_1 - x_2)^2}{2}$$
$$= (x_1 + x_2 + x_3)^2 - 3(x_2x_3 + x_3x_1 + x_1x_2) = -9C.$$

Hence (2.3) goes over into

$$x_1 = \tfrac{2}{3}q\cos\Theta, \quad x_2 = \tfrac{2}{3}q\cos(\Theta + 120°), \quad x_3 = \tfrac{2}{3}q\cos(\Theta + 240°), \tag{2.8}$$

[2] Such a proof has been suggested by Beatty (in a letter to the editor): Since for a maximum or minimum of Δ^2 in the sense of differential calculus the jacobian

$$\frac{\partial(s, H, \Delta^2)}{\partial(a, b, c)}$$

must vanish, from (1.2), (1.7) and (1.15) it would follow rather immediately that the triangle is isosceles.

and from (2.4) and (2.7) it follows that

$$D = -\frac{2}{27}q^3\cos 3\Theta. \tag{2.9}$$

Let us now express Δ^2 as a function of q, s, and Θ. We have:

$$\frac{\Delta^2}{s} = (s-a)(s-b)(s-c) = \left(\frac{s}{3}-x_1\right)\left(\frac{s}{3}-x_2\right)\left(\frac{s}{3}-x_3\right), \tag{2.10}$$

and from the identity

$$(x-x_1)(x-x_2)(x-x_3) = x^3 + 3Cx + D \tag{2.11}$$

it follows for $x = \frac{1}{3}s$ that

$$\frac{\Delta^2}{s} = \frac{s^3 + 27Cs + 27D}{27}; \tag{2.12}$$

finally, using the values of C and D given by (2.7) and (2.9), we are led to the formula[3]

$$\Delta^2 = \frac{s}{27}(s^3 - 3sq^2 - 2q^3\cos 3\Theta). \tag{2.13}$$

From (2.13) it is now obvious that the maximum of Δ^2 is

$$\frac{s}{27}(s^3 - 3sq^2 + 2q^3) = \frac{s(s-q)^2(s+2q)}{27}, \tag{2.14}$$

and it is really attained by taking $\Theta = 60°$, i.e.—see (2.5) and (2.8)—for an isosceles triangle with sides

$$a = c = \frac{2s+q}{3}, \quad b = \frac{2(s-q)}{3}. \tag{2.15}$$

The minimum of Δ^2, corresponding to the second equality sign in (1.3), viz.

$$\frac{s(s^3 - 3sq^2 - 2q^3)}{27} = \frac{s(s+q)^2(s-2q)}{27}, \tag{2.16}$$

is obtained by taking $\Theta = 0$ in (2.13), i.e. for an isosceles triangle with sides

$$a = \frac{2(s+q)}{3}, \quad b = c = \frac{2s-q}{3}, \tag{2.17}$$

if that triangle exists, i.e., if the conditions (1.13) are fulfilled. From (2.17) it is immediately seen that this is only the case if $q < \frac{1}{2}s$; but it has already been mentioned above (§1) that there is no positive minimum of Δ^2 in the exceptional case $q \geqslant \frac{1}{2}s$ (i.e., $5H \geqslant 3K$ in Beatty's notation).

Finally it should be remarked that the use of trigonometric functions in the

[3]Professor Gordon Pall has pointed out that the following more general formula can be proved along the same lines:

$$(k-a)(k-b)(k-c) = \tfrac{8}{27}q^3(p^3 - \tfrac{3}{4}p - \tfrac{1}{4}\cos 3\Theta), \text{ where } p = (3k-2s)/2q.$$

foregoing proof might be avoided by proving instead of (2.13) the following algebraic identity:

$$\left\{\Delta^2 - \frac{s^2}{27}(s^2 - 3q^2)\right\}^2 = \left(\frac{s}{27}\right)^2\left\{4q^6 - 27(b-c)^2(c-a)^2(a-b)^2\right\} \tag{2.18}$$

and obtaining thus the following inequality equivalent to (1.3):

$$\left|\Delta^2 - \frac{s^2}{27}(s^2 - 3q^2)\right| < \frac{2}{27}sq^3. \tag{2.19}$$

3. Bounds for the product *abc*. In virtue of (1.15), one might obtain from (1.3), or from (2.15) and (2.17), exact bounds also for the product *abc*:

$$\frac{2}{27}(s-q)(2s+q)^2 < abc < \frac{2}{27}(s+q)(2s-q)^2, \tag{3.1}$$

giving us the solution of the following algebraic problem which might be of some interest in itself: to find maximum and minimum of the product of three positive variables a, b, c, if two of their symmetric functions, viz. s and q, are given. Since

$$q^2 = 4s^2 - 3K, \tag{3.2}$$

it is easy to obtain from (3.1) the exact bounds for *abc* also in the case when the first two *elementary* symmetric functions, viz. $2s$ and K, are given:

$$\frac{2}{27}\left\{s(9K - 8s^2) - (4s^2 - 3K)^{3/2}\right\} < abc < \frac{2}{27}\left\{s(9K - 8s^2) + (4s^2 - 3K)^{3/2}\right\}. \tag{3.3}$$

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THE MINIMUM OF A DEFINITE TERNARY QUADRATIC FORM

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A purely algebraic solution for the problem of the extreme definite ternary quadratic form has been given by Mordell **(2)**. The equivalent geometric problem of the critical lattice of a sphere was solved by Mrs. Ollerenshaw **(3)** under a preliminary assumption that was said to be justified by "simple considerations" (p. 297). The present treatment eliminates this weakness, and avoids the use of trigonometric functions, without adding undue complications. It arose from a remark of Hilbert and Cohn-Vossen **(1**, p. 45**)** which states that their method for the analogous two-dimensional problem **(1**, pp. 35–37**)** can be generalized to any number of dimensions. However, the generalization is not trivial, even in three dimensions, and the four-dimensional treatment of Mrs. Ollerenshaw **(4)** involves a preliminary assumption analogous to that mentioned above.

We begin by briefly recapitulating Gauss's statement of the connection between the algebraic and geometric problems. Three independent vectors $\mathbf{t}_r$ $(r = 1, 2, 3)$ generate a system of vectors $u_r\mathbf{t}_r$ (u_r integers) which lead from the origin to the points of a lattice. We may regard (u_1, u_2, u_3) as affine coordinates for the lattice points. The square of the distance from (0, 0, 0) to (u_1, u_2, u_3) is

$$(\sum u_r\mathbf{t}_r)^2 = \sum\sum a_{rs}\, u_r\, u_s,$$

where $a_{rs} = \mathbf{t}_r \cdot \mathbf{t}_s$. Thus the lattice determines a positive definite ternary quadratic form whose minimum m is the square of the minimum distance c between lattice points. If the vector $\mathbf{t}_r$ has Cartesian components (x_r, y_r, z_r), the determinant of the form is

$$\begin{aligned} D = \det(\mathbf{t}_r \cdot \mathbf{t}_s) &= \det(x_r x_s + y_r y_s + z_r z_s) \\ &= \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = V^2, \end{aligned}$$

where V is the volume of the elementary cell (or primitive parallelepiped) of the lattice. Since

$$\frac{D}{m^3} = \left(\frac{V}{c^3}\right)^2,$$

the form of smallest determinant for its minimum value corresponds to the lattice of smallest cell for its minimum distance. Such a lattice we now seek.

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Consider any point A of a given lattice whose elementary cell has volume V. Choose a lattice point B at the minimum distance c from A, and a lattice point C outside the line AB, at the shortest distance b ($\geqslant c$) from A. These points can always be chosen so that $\angle CAB \leqslant \frac{1}{2}\pi$, and the sides $a = BC$, $b = CA$, $c = AB$ of the triangle ABC satisfy

$$a \geqslant b \geqslant c, \quad a^2 \leqslant b^2 + c^2.$$

Let Δ and R denote the area and circumradius of this triangle, so that

$$16\Delta^2 = -a^4 - b^4 - c^4 + 2b^2c^2 + 2c^2a^2 + 2a^2b^2, \qquad 4R\Delta = abc.$$

In a parallel lattice plane nearest to the plane ABC, there is a lattice point D whose orthogonal projection D_1 on the plane ABC lies inside or on a side of the parallelogram $ABA'C$, and so, by choice of notation, inside or on a side of the triangle ABC. Denote by d the distance DD_1 from D to the plane ABC, so that

$$V = 2\Delta d.$$

Since none of AD, BD, CD is parallel to AB, all of them are greater than or equal to b. Since the triangle ABC has no obtuse angle, circles of radius R with centres at the vertices overlap in such a way that every point of the triangle except the circumcentre is inside at least one of them (**3**, p. 298, footnote). Therefore the distance of D_1 from at least one vertex is less than R, except that it is equal to R when D_1 is the circumcentre. Thus at least one of AD, BD, CD is less than or equal to $(R^2 + d^2)^{\frac{1}{2}}$, and consequently

$$b^2 \leqslant R^2 + d^2,$$

with equality only when D_1 is the circumcentre. Hence

$$\begin{aligned} V^2 &= (2\Delta d)^2 \geqslant 4\Delta^2(b^2 - R^2) \\ &= \tfrac{1}{4}b^2(-a^4 - b^4 - c^4 + 2b^2c^2 + c^2a^2 + 2a^2b^2) \\ &= \tfrac{1}{2}c^6 + \tfrac{1}{4}c^2(b^2 - c^2)(3b^2 + 2c^2) + \tfrac{1}{4}b^2(a^2 - b^2)(b^2 + c^2 - a^2) \\ &\geqslant \tfrac{1}{2}c^6, \end{aligned}$$

with equality only when

$$d^2 = b^2 - R^2, \quad b = c,$$

and either

$$\text{(i) } a = b \quad \text{or} \quad \text{(ii) } b^2 + c^2 = a^2.$$

These conditions (i) and (ii) are sufficient to determine two "critical" lattices Λ_1 and Λ_2 for either of which D/m^3 or $(V/c^3)^2$ attains its minimum value $\frac{1}{2}$.

In case (i), the tetrahedron $ABCD$ is regular, and we may choose Cartesian coordinates

$$(0, 0, 0), \quad (0, 1, 1), \quad (1, 0, 1), \quad (1, 1, 0)$$

for its vertices. Thus

$$\mathbf{t}_1 = AB = (0, 1, 1), \quad \mathbf{t}_2 = AC = (1, 0, 1), \quad \mathbf{t}_3 = AD = (1, 1, 0)$$

and

$$\begin{aligned}(\sum u_r \mathbf{t}_r)^2 &= (u_2 + u_3, u_3 + u_1, u_1 + u_2)^2 \\ &= (u_2 + u_3)^2 + (u_3 + u_1)^2 + (u_1 + u_2)^2 \\ &= 2(u_1^2 + u_2^2 + u_3^2 + u_2 u_3 + u_3 u_1 + u_1 u_2).\end{aligned}$$

In case (ii), the triangle ABC is right-angled isosceles. Choosing A, B and D as before, we now have C at $(0, 1, -1)$. Thus

$$\mathbf{t}_1 = AB = (0, 1, 1), \quad \mathbf{t}_2 = AC = (0, 1, -1), \quad \mathbf{t}_3 = AD = (1, 1, 0)$$

and

$$\begin{aligned}(\sum u_r \mathbf{t}_r)^2 &= (u_3, u_1 + u_2 + u_3, u_1 - u_2)^2 \\ &= u_3^2 + (u_1 + u_2 + u_3)^2 + (u_1 - u_2)^2 \\ &= 2(u_1^2 + u_2^2 + u_3^2 + u_2 u_3 + u_3 u_1).\end{aligned}$$

In either case, the lattice generated by the $\mathbf{t}$'s consists of all points with integral coordinates whose sum is even. Thus the lattices Λ_1 and Λ_2 are the same, viz., the face-centred cubic lattice (**1**, pp. 46, 56). (The points whose coordinates are all even form an ordinary cubic lattice, while those which have two odd coordinates are the centres of the square faces.) Since the two sets of $\mathbf{t}$'s are different bases for the same lattice, the two forms are equivalent: there is only one class of extreme ternary forms.

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THE PERFECT AND EXTREME SENARY FORMS

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This paper is devoted partly to a summary of some results which will be published in detail elsewhere,[1] partly to a description of a new extreme senary form, ϕ_6, and an n-dimensional generalization of it.

The main result is the determination of all classes of extreme senary forms. There are just six classes, represented by:

$$\phi_0 = \sum_1^6 x_i^2 + \sum_{i<j} x_i x_j,$$
$$\phi_1 = \phi_0 - x_1x_2,$$
$$\phi_2 = \phi_0 - x_1x_2 - x_1x_3,$$
$$\phi_3 = \phi_0 - \tfrac{1}{2}(x_1x_2 + x_3x_4 + x_5x_6),$$
$$\phi_4 = \phi_0 - \tfrac{1}{2}(x_1x_2 + x_3x_4 + x_3x_5 + x_3x_6 + x_4x_5 + x_4x_6 + x_5x_6),$$
$$\phi_5 = \phi_0 - \tfrac{1}{2}(2x_1x_2 + x_1x_3 + x_1x_6 + x_2x_5 + x_4x_6 + 2x_5x_6).$$

Each of these forms has minimum $M = 1$ for integral $\mathbf{x} \neq \mathbf{0}$. They are listed in the following table, together with the symbol used by Coxeter **(3)**, the order of the group $\mathfrak{g}$ of automorphs, the number s of pairs of minimal vectors, and the value of $2^6D/M^6$ in decreasing order of this quantity.

Form	Coxeter's symbol	Order of $\mathfrak{g}$	s	$2^6D/M^6$
ϕ_0	A_6	2.7!	21	7
ϕ_3		96	21	$13.3^3/2^6$
ϕ_6		672	21	$7^3/2^6$
ϕ_1	$B_6 \sim D_6$	2^6.6!	30	4
ϕ_4	E_6^3	144.6!	27	$3^3/2^6$
ϕ_2	E_6	144.6!	36	3

Thus there is just one class of absolutely extreme forms, represented by ϕ_2; and, in agreement with Blichfeldt **(2)**,

$$\gamma_6^6 = \max_f \frac{M^6}{D} = \frac{1}{D(\phi_2)} = \frac{64}{3}.$$

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[1] *The complete enumeration of extreme senary forms*, Phil. Trans. Roy. Soc. London, *A* 249 (1957), 461-506.

The method used to establish these forms is the algorithm for perfect forms given by Voronoi **(6)**. Hence all classes of perfect senary forms are now determined; these are represented by the extreme forms $\phi_0, \ldots, \phi_4, \phi_6$ and

$$\phi_5 = \phi_0 - \tfrac{1}{2}(x_1x_2 + x_3x_4 + x_3x_5 + x_4x_5).$$

The forms ϕ_0, ϕ_1, ϕ_2 and ϕ_4 were found by Coxeter **(3)**, who gives a full description of their properties. Previously, both Voronoi **(6)** and Hofreiter **(4)** had established ϕ_0, ϕ_1 and ϕ_2 (this being Voronoi's notation for them). As is pointed out by Coxeter **(3)**, Hofreiter's list contains a fourth form, F_4, which is not extreme, but which is presumably an arithmetical slip for[2] ϕ_3.

The form ϕ_3 was recently discovered independently by Kneser **(5)** and the author **(1)**, and the reader is referred to these articles for a full description of its properties.

The remaining extreme form ϕ_6 and the perfect non-extreme form ϕ_5 are new. Apart from its existence as the simplest known form of its type, ϕ_5 has little structural interest; we merely note that it has

$$M = 1, \quad D = 3^4/2^{10}, \quad s = 22,$$

and that its group $\mathfrak{g}$ is of order 288.

On applying to $\phi_6(\mathbf{x})$ the transformation

$$\mathbf{x} = \tfrac{1}{7}\begin{pmatrix} 3 & 6 & 2 & 5 & 1 & 4 \\ 5 & 3 & 1 & 6 & 4 & 2 \\ 4 & 1 & 5 & 2 & -1 & 3 \\ -4 & -1 & 2 & -2 & 1 & -3 \\ -1 & -2 & -3 & -4 & 2 & 1 \\ -6 & -5 & -4 & -3 & -2 & -1 \end{pmatrix}\mathbf{y},$$

we obtain

$$4\phi_6(\mathbf{x}) = 2\sum_1^6 y_i^2 + 2\sum_{i<j} y_iy_j = 2\phi_0(\mathbf{y}); \tag{2}$$

or, defining y_7 by the relation

$$\sum_1^7 y_i = 0, \tag{3}$$

$$4\phi_6(\mathbf{x}) = \sum_1^7 y_i^2. \tag{4}$$

Since $2\phi_0$ has determinant 7, and the transformation (1) has determinant $1/7$, it follows that $D(\phi_6) = 7^3/2^{12}$. It is also easily verified from (1) that $\mathbf{x}$ is integral if and only if $\mathbf{y}$ is integral and satisfies

$$\sum_1^7 iy_i \equiv 0 \pmod 7. \tag{5}$$

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[2]As Professor Coxeter has remarked to me, the correct value of $2^8D/M^6$ for ϕ_3 is obtained by replacing 53 by 52 in F_4.

Thus ϕ_6 now appears as the case $n = 6$ of the form f_n defined, in terms of its lattice, by

$$f_n = \sum_1^m y_i^2, \tag{6}$$

where the y_i take integral values subject to

$$\sum_1^m y_i = 0, \tag{7}$$

$$\sum_1^m i y_i \equiv 0 \pmod{m} \tag{8}$$

(where we have set $m = n + 1$ for convenience). This form is extreme for all $n > 6$, but the proof of this for odd n is rather involved. In what follows we shall restrict ourselves to even n (odd m).

Let $\mathbf{e}_1, \ldots, \mathbf{e}_m$ denote the unit vectors in m-space, with coordinates $y_1, \ldots, y_m$. Now f_n takes only even values subject to (7). It takes the value 2 only when $\mathbf{y} = \mathbf{e}_i - \mathbf{e}_j$, and clearly none of these sets satisfies (8). It takes the value 4 when

$$\mathbf{y} = \mathbf{e}_a + \mathbf{e}_b - \mathbf{e}_c - \mathbf{e}_d \tag{9}$$

with

$$a + b \equiv c + d \pmod{m}. \tag{10}$$

Thus $M(f_n) = 4$. Since the lattice defined by (7), (8) has determinant $m^{\frac{1}{2}} \cdot m$, we have $D(f_n) = m^{\frac{3}{2}}$; so

$$(2/M)^n D = (n + 1)^3/2^n.$$

The number of pairs of minimal vectors is[3]

$$s = \tfrac{1}{8} n(n + 1)(n - 2) \qquad (n \text{ even}).$$

For, for any a, b, the number of distinct unordered pairs c, d satisfying (10) is $\frac{1}{2}(m - 1)$, of which one pair is a, b. Since a, b may be chosen in $\frac{1}{2}m(m - 1)$ ways, this gives $2s = \frac{1}{2}m(m - 1)\frac{1}{2}(m - 3)$, as asserted.

We now show that f_n is eutactic. Regarding y_m as the redundant variable in (6), we have

$$f_n = \left(\sum_1^n y_i\right)^2 + \sum_1^n y_i^2,$$

with adjoint

$$F_n = n \sum_1^n y_i^2 - 2 \sum_{i<j} y_i y_j.$$

The linear forms associated with the minimal vectors (9) are then

$$\lambda_k = y_a + y_b - y_c - y_d \qquad (k = 1, \ldots, s), \tag{11}$$

[3]For odd n, $s = \frac{1}{8}(n - 1)^3(n + 1)$.

with $y_m = 0$. We now see that

$$\tfrac{1}{2}(m-3)F_n = \sum_1^s \lambda_k^2. \tag{12}$$

For, by symmetry, each y_i occurs in $4s/m = \frac{1}{2}(m-1)(m-3)$ forms λ_k, giving a coefficient $\frac{1}{2}(m-1)(m-3)$ of y_i^2 in $\sum\lambda_k^2$. A pair y_i, y_j occurs with like signs in $\frac{1}{2}(m-3)$ forms (as above); and with opposite signs in $m-3$ forms, this being the number of solutions of $d - b \equiv i - j$ with d, b distinct from i and j. This gives a coefficient $m - 3 - 2(m-3) = -(m-3)$ of y_iy_j in $\sum\lambda_k^2$. Collecting these results and setting $y_m = 0$, we obtain (12).

To prove that f_n is perfect, we begin by defining the vectors

$$\mathbf{f}_i = \mathbf{e}_i + \mathbf{e}_{i+1} - \mathbf{e}_{i+2} - \mathbf{e}_{i-1} \qquad (i = 1, \ldots, m),$$

where all suffixes and congruences are to be interpreted modulo m. Then

$$\sum_1^m \mathbf{f}_i = 0, \tag{13}$$

but any $m-1$ of the $\mathbf{f}_i$ are independent. Hence we may take a new coordinate system $z_1, \ldots, z_n$ defined by

$$\mathbf{z} = \sum_1^n z_i\mathbf{f}_i.$$

We have now to show that if all minimal vectors $\mathbf{z}$ of f_n satisfy a quadratic relation

$$p(\mathbf{z}) = \sum_1^n p_{ij}z_iz_j = 0, \tag{14}$$

then all $p_{ij} = 0$. Using (12), we shall express some of the minimal vectors in terms of $\mathbf{f}_1, \ldots, \mathbf{f}_m$; if $\mathbf{f}_m$ occurs in the expression, we must of course replace it by

$$-\sum_1^n \mathbf{f}_i$$

before using (14).

(i) Suppose first that $m \geqslant 9$ (and odd). We use in (14) the minimal vectors

$$\mathbf{f}_i + \mathbf{f}_{i+1} + \ldots + \mathbf{f}_{i+r} = \mathbf{e}_{i+r} + \mathbf{e}_{i+1} - \mathbf{e}_{i+r+2} - \mathbf{e}_{i-1} \qquad (r \not\equiv \pm 1, -3) \tag{15}$$

$$\mathbf{f}_i + \mathbf{f}_{i+2} + \ldots + \mathbf{f}_{i+2r} = \mathbf{e}_i + \mathbf{e}_{i+2r+1} - \mathbf{e}_{i+2r+2} - \mathbf{e}_{i-1} \qquad (2r \not\equiv -1, -2, -3), \tag{16}$$

$$-\mathbf{f}_{i-3} + \mathbf{f}_i + \mathbf{f}_{i+1} + \ldots + \mathbf{f}_{i+m-6} = \mathbf{e}_{i-6} + \mathbf{e}_{i+1} - \mathbf{e}_{i-3} - \mathbf{e}_{i-2}, \tag{17}$$

the given conditions ensuring that the suffixes on the right are distinct. Values of i giving a suffix $0 (\mathrm{mod}\ m)$ on the left will not be considered.

We obtain at once from (15), with $r = 0$,

$$p(\mathbf{f}_i) = p_{ii} = 0. \tag{18}$$

Using (15) with indices $r, r-1$, and subtracting, we obtain

$$2p_{i,i+r} + 2p_{i+1,\,i+r} + \ldots + 2p_{i+r-1,\,i+r} + p_{i+r,\,i+r} = 0;$$

hence, from (18),

$$p_{i,i+r} + p_{i+1,\,i+r} + \ldots + p_{i+r-1,\,i+r} = 0 \qquad (r = 3, \ldots, m-4). \tag{19}$$

Replacing i, r by $i+1, r-1$ and subtracting from (19) gives

$$p_{i,i+r} = 0 \qquad (r = 4, \ldots, m-4). \tag{20}$$

From (16) with $r = 1$, and $p_{ii} = 0$,

$$p_{i,i+2} = 0, \quad p_{1,m-1} = 0; \tag{21}$$

from (15) with $r = 2$ and (21),

$$p_{i,i+1} + p_{i+1,\,i+2} = 0. \tag{22}$$

From (19) with $r = 3$ and (21)

$$p_{i,i+3} + p_{i+2,\,i+3} = 0. \tag{23}$$

We now use (17), and (15) with $r = m-6$; subtracting the results gives

$$p_{i-3,i} + p_{i-3,\,i+1} + \ldots + p_{i-3,\,i+m-6} = 0.$$

This holds certainly for $i = 4$ and $i = 1$ (no suffix then being zero), whence

$$p_{14} + p_{15} + \ldots + p_{1,m-2} = 0,$$
$$p_{1,m-2} + p_{2,m-2} + \ldots + p_{m-5,\,m-2} = 0.$$

By (20), these reduce to

$$p_{14} + p_{1,m-2} = p_{1,m-2} + p_{m-5,\,m-2} = 0,$$

whence

$$p_{14} = p_{m-5,\,m-2}.$$

By (23) with $i = 1$ and $i = m-5$, this gives

$$p_{34} = p_{m-3,\,m-2}. \tag{24}$$

But (22) gives

$$p_{34} = -p_{45} = p_{56} = \ldots = -p_{m-3,\,m-2},$$

and so, with (24),

$$p_{34} = p_{m-3,\,m-2} = 0.$$

Now (22) and (23) give at once

$$p_{i,i+1} = 0, \quad p_{i,i+2} = 0.$$

This, with (18), (20) and (21), establishes that all $p_{ij} = 0$ except possibly $p_{1,m-2}$ and $p_{2,m-1}$. That these are also zero now follows immediately by using the minimal vectors (from (15) with $r = 1$)

$$-(\mathbf{f}_m + \mathbf{f}_2) = \mathbf{f}_1 + \mathbf{f}_3 + \mathbf{f}_4 + \ldots + \mathbf{f}_{m-1},$$
$$-(\mathbf{f}_{m-2} + \mathbf{f}_m) = \mathbf{f}_1 + \mathbf{f}_2 + \ldots + \mathbf{f}_{m-3} + \mathbf{f}_{m-1}.$$

Thus all $p_{ij} = 0$, and so f_n is perfect.

(ii) When $m = 7$, (17) is no longer a minimal vector and the above analysis fails. All 21 vectors are now

$$\mathbf{f}_i, \mathbf{f}_i + \mathbf{f}_{i+2}, \mathbf{f}_i + \mathbf{f}_{i+1} + \mathbf{f}_{i+2} \qquad (i = 1, \ldots, 7), \tag{25}$$

and a simple calculation establishes the perfection of $f_n = \phi_6$.

It is easy to verify that $\mathbf{f}_1, \ldots, \mathbf{f}_{m-1}$ give not only a basis of the space, but also a basis for the lattice of f_n. Hence we find the cyclically symmetrical representation

$$f_n(\mathbf{x}) = \sum_1^m (x_i + x_{i-1} - x_{i-2} - x_{i+1})^2 \qquad (n \text{ even}),$$

with $x_m = 0$, the suffixes being taken modulo m. A similar representation exists also for n odd.

We now consider the group $\mathfrak{g}$ of automorphs of ϕ_6; we denote by $\mathfrak{G}$ the contragadient group of automorphs of its adjoint F_6. Since $\mathfrak{g}$ must permute the minimal vectors (more precisely, the pairs $\pm\mathbf{m}$ of minimal vectors), $\mathfrak{G}$ permutes the associated linear forms. From (11), any permutation of the λ_k leaves F_6 invariant. It follows that $\mathfrak{G}$ is precisely the group of linear transformations permuting the associated linear forms.

From this result, we shall show that $\mathfrak{G}$ (and so $\mathfrak{g}$) has order 672, is transitive on the linear forms,[4] and is isomorphic to $\mathfrak{C}_2 \times PGL(2, 7)$. Corresponding to the representation (25), we may write the associated linear forms (11) as

$$z_i, z_i + z_{i+2}, z_i + z_{i+1} + z_{i+2} \qquad (i = 1, \ldots, 7),$$

where

$$z_i = y_i + y_{i+1} - y_{i+2} - y_{i-1},$$
$$\sum_1^m z_i = 0.$$

Obvious elements of $\mathfrak{G}$ are S: $y_i \to y_{i+1}$ and U: $y_i \to y_{3i}$ (where the suffixes are taken modulo 7). These show that $\mathfrak{G}$ is transitive on the linear forms. For $Sz_i = z_{i+1}$, so that the forms of each set (z_i), $(z_i + z_{i+2})$, $(z_i + z_{i+1} + z_{i+2})$ are permuted cyclically by S; while

$$Uz_7 = y_7 + y_3 - y_6 - y_4 = z_7 + z_2,$$
$$U^2z_7 = y_7 + y_2 - y_4 - y_5 = -(z_3 + z_4 + z_5).$$

Let now $\mathfrak{G}_7$ be the subgroup of $\mathfrak{G}$ which leaves z_7 invariant (or changes its sign). We shall say that two of the linear forms *combine* if their sum or difference is an associated linear form. This property being invariant under $\mathfrak{G}$, $\mathfrak{G}_7$ must permute the 8 forms combining with z_7, namely

$$z_1 + z_2 + z_3, \quad -z_3 - z_7, \quad z_6 + z_1, \quad -z_7 - z_2, \tag{26}$$
$$z_4 + z_5 + z_6, \quad z_4, \quad -z_6 - z_7 - z_1, \quad z_2.$$

[4] Contrary to what one might expect from (12), $\mathfrak{G}$ is no longer transitive for $n > 6$.

Using the element T of $\mathfrak{G}_7$, of order 8, defined by

$$T(z_1, \ldots, z_7) = (-z_7 - z_1 - z_2, \\ z_1 + z_2 + z_3, -z_3 - z_5, z_5 + z_6 + z_7, -z_6 - z_7 - z_1, z_1, z_7)$$

we see that $\mathfrak{G}_7$, considered as a permutation group on the set (26), is transitive on this set. The successive elements $T^n z_2 (n = 1, \ldots, 8)$ are in fact the 8 elements (26), in the order written.

Let then $\mathfrak{G}_{7,2}$ be the subgroup of $\mathfrak{G}_7$ which leaves z_2 invariant. (We note that $\mathfrak{G}_{7,2}$ must leave the relative signs of z_7, z_2 unaltered, since $z_7 + z_2$ is a linear form, while $z_7 - z_2$ is not.) $\mathfrak{G}_{7,2}$ must permute the 2 forms $z_6 + z_7 + z_1$, $z_1 + z_2 + z_3$ which combine with both z_7 and z_2. A direct argument now shows that $\mathfrak{G}_{7,2}$ is of order 4 and consists of the elements $\pm I$, $\pm V$, where

$$V(z_1, \ldots, z_7) = (-z_7 - z_1 - z_2, z_2, -z_6, -z_2 - z_4, -z_5 - z_7, -z_3, z_7).$$

It follows that $\mathfrak{G}$ has order $21.8.4 = 672$, as asserted, and is generated by S, T, U, V and $-I$. Since

$$V = -S^2T^6S, \quad U = -T^6S^5T^7,$$

we see that $\mathfrak{G} = \{S, T, -I\} = \{-I\} \times \{S, T\}$.

We may identify $\{S, T\}$ with $PGL(2, 7)$, consisting of the regular 2×2 matrices over $GF(7)$ (where multiples of the same matrix are identified), by means of the mapping

$$S \leftrightarrow \sigma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$T \leftrightarrow \tau = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}.$$

For, as is well known, $PSL(2, 7)$ (consisting of those elements of $PGL(2, 7)$ whose determinant is a square) is a simple group of order 168 generated by σ and

$$\theta = \tau^4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and may be defined by the relations

$$\sigma^7 = \theta^2 = (\sigma\theta)^3 = (\sigma^4\theta)^4 = 1. \tag{27}$$

Now τ has determinant 5, a non-square, so that σ, τ generate $PGL(2, 7)$; we may take the further defining relation as

$$\tau^{-1}\sigma\tau = \sigma^4\theta\sigma^2\theta\sigma^5. \tag{28}$$

A simple calculation shows that the relations (27), (28) are satisfied if σ, τ, θ are replaced by S, T, T^4 respectively. We note that, under the given mapping, the elements of $PSL(2, 7)$ correspond to the elements of $\mathfrak{G}$ of determinant $+1$ (expressed as integral unimodular transformations on, say, $z_1, \ldots, z_6$).

We consider finally a conjecture made by Coxeter (3, p. 392), that every eutactic form with $s \geqslant \frac{1}{2}n(n+1)$ is perfect, and so extreme (the condition $s \geqslant \frac{1}{2}n(n+1)$ being of course *necessary* for perfection). We can use the form f_n considered above to show that the conjecture is unfortunately false.

Let $f(x_1, \ldots, x_n)$ be any eutactic form with $s \geqslant \frac{1}{2}n(2n+1)$; f_n is such a form for even $n \geqslant 10$ (in fact for all $n > 10$). Define

$$g(x_1, \ldots, x_{2n}) = f(x_1, \ldots, x_n) + f(x_{n+1}, \ldots, x_{2n}).$$

Then $M(g) = M(f)$, and is attained when and only when one f is zero and the other assumes its minimum. Hence the number of pairs of minimal vectors of g is $2s \geqslant \frac{1}{2} \cdot 2n(2n+1)$. Since f is eutactic and the adjoint of g is a multiple of $F(x_1, \ldots, x_n) + F(x_{n+1}, \ldots, x_{2n})$, g is clearly eutactic. But g is not perfect, since it is disconnected: all minimal vectors satisfy, for example, the relation $x_1 x_{n+1} = 0$.

I am indebted to Dr. G. E. Wall for helpful discussions on the group of ϕ_8.

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GROUPS GENERATED BY UNITARY REFLECTIONS OF PERIOD TWO

H. S. M. COXETER

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1. Introduction. In complex affine n-space with a unitary metric, a *reflection* is a congruent transformation leaving invariant all the points of a hyperplane. Thus the characteristic roots of a unitary reflection of period p consist of a primitive pth root of unity and $n - 1$ unities. A group generated by n reflections is conveniently represented by a graph having a node for each generator and a branch for each pair of non-commutative generators. For a generator of period p, the node is generally marked p, but we find it convenient to omit the mark when $p = 2$, as in the case of real reflections **(8**, p. 619**)**. Whenever two such involutory generators do not commute, the corresponding nodes are joined by a branch which is marked with the period of their product, except that for simplicity the mark is omitted when this period takes its most prevalent value 3.

Nodes representing commutative generators are not (directly) joined. This convention has the happy result that the graph for a (completely) reducible group consists of disconnected pieces representing the irreducible components.

For any finite irreducible group generated by n involutory reflections, Shephard **(29)** and Todd **(31)** showed, in effect, that the generators may be so chosen that the graph either is a tree or contains just one triangular circuit. Since the abstract group defined by

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$$R_1^2 = R_2^2 = R_3^2 = (R_2R_3)^{l_1} = (R_3R_1)^{l_2} = (R_1R_2)^{l_3} = E \qquad (l_1, l_2, l_3 > 2)$$

is infinite **(32**, pp. 26–29**)**, the three generators represented by the nodes of such a circuit must satisfy some further relation. One of the purposes of the present paper is to indicate the advantages of using the relation

$$(R_1R_2R_3R_2)^m = E,$$

which is equivalent to $(R_2R_3R_1R_2)^m = E$ or $(R_3R_1R_2R_1)^m = E$ if, as can always be arranged, at least two of the l's take the value 3. Accordingly, we complete the graphical symbol by writing m inside the triangle. This agrees with the notation of Shephard **(29**, 370–374**)** when $m = 3$; for other values of m there is an essential difference, as we shall see.

The finite irreducible groups generated by three real reflections are the symmetry groups of the Platonic solids **(26a**, pp. 20, 24**)**. We shall exhibit all the remaining finite three-dimensional irreducible groups generated by three reflections as instances of $[1\ 1\ 1^l]^m$ (see 3.8), which is finite when l and m satisfy the inequality 3.7. In particular, $[1\ 1\ 1^4]^4$, which is "No. 24" in the list of Shephard and Todd **(31**, p. 301**)**, is interesting because of its connection with the new senary extreme form discovered by Barnes (see the preceding paper **(4)**).

Groups in more than three dimensions are derived by adding "tails" to the graph for $[1\ 1\ 1^l]^m$. The graph so obtained enables us to compute certain numbers $m_1, m_2, \ldots, m_n$ which quickly yield both the order of the group and the order of its centre.

Reflections of period $p > 2$ will be discussed in a subsequent paper, where it will be shown that the proper interpretation for a branch marked 4 or 5 is not

$$(R_1R_2)^4 = E \qquad \text{or} \qquad (R_1R_2)^5 = E$$

but

$$R_1R_2R_1R_2 = R_2R_1R_2R_1 \text{ or } R_1R_2R_1R_2R_1 = R_2R_1R_2R_1R_2.$$

2. Reflections in the coordinate hyperplanes. In complex affine n-space, with the "contravariant" notation $(x^1, \ldots, x^n)$ for coordinates, any finite group of affine collineations (i.e., linear transformations) leaves invariant a positive definite Hermitian form, say

2.1 $$\sum\sum a_{jk} x^j \bar{x}^k \qquad (a_{jk} = \bar{a}_{kj})$$

(6, p. 253**)**. This form determines a unitary metric in which the point $(x^1, \ldots, x^n)$ may be usefully regarded as having also covariant coordinates $(x_1, \ldots, x_n)$, defined by

2.2 $$x_j = \sum a_{jk} \bar{x}^k \qquad (j = 1, \ldots, n).$$

In this notation, the form 2.1 may be expressed either as

2.3 $$\sum x_j x^j$$

or as

2.4 $$\sum\sum a^{jk}\, x_j\, \bar{x}_k,$$

where the new coefficients $a^{jk} = \bar{a}^{kj}$ are given by

$$\sum a_{hk}\, a^{jk} = \delta_h^j,$$

so that

2.5 $$x^j = \sum a^{jk}\, \bar{x}_k, \quad \bar{x}^k = \sum a^{jk}\, x_j.$$

Embedding the affine n-space in a projective n-space, we may define an *affine reflection* (**36**, pp. 109, 115) to be a homology whose centre is a point at infinity, and define a *unitary reflection* (**28**, p. 82) to be an affine reflection that leaves invariant the form 2.1 or 2.3 or 2.4.

For a group generated by n unitary reflections, we may choose such a frame of reference that the centres of the n homologies are the points at infinity on the contravariant axes. Then R_k, the kth generating reflection, leaves invariant all the contravariant coordinates x^j except x^k. Let us suppose that it transforms x^k into

$$\sum c_l\, x^l.$$

Since 2.1 is invariant, we have

$$a_{kk}\, x^k\, \bar{x}^k + \sum_{j\neq k} (a_{jk}\, x^j\, \bar{x}^k + a_{kj}\, x^k\, \bar{x}^j)$$
$$= a_{kk} \sum c_l\, x^l \sum \bar{c}_l\, \bar{x}^l + \sum_{j\neq k} (a_{jk}\, x^j \sum \bar{c}_l\, \bar{x}^l + a_{kj}\, \bar{x}^j \sum c_l\, x^l),$$

whence, by comparing coefficients of $x^j\, \bar{x}^k$ $(j \neq k)$,

2.6 $$a_{jk} = a_{kk}\, c_j\, \bar{c}_k + a_{jk}\, \bar{c}_k.$$

In the present paper we restrict consideration to cases where R_k is involutory (i.e., of period 2), so that

$$c_k = -1$$

and therefore $2a_{jk} + a_{kk}\, c_j = 0$ $(j \neq k)$. Since the vanishing of a_{kk} would imply $a_{jk} = 0$ for all j, we must have $a_{kk} \neq 0$ for each k. Thus

2.7 $$c_j = -2a_{jk}/a_{kk} \qquad (j \neq k),$$

and R_k is the transformation leaving invariant every x^j except x^k, which becomes

2.8 $$\sum c_j\, x^j = x^k - \frac{2\sum a_{jk}\, x^j}{a_{kk}} = x^k - \frac{2\bar{x}_k}{a_{kk}}$$

(cf. **17**, p. 403). Thus R_k leaves invariant every point on the hyperplane

$$x_k = 0.$$

In other words, the reflecting hyperplanes are the covariant coordinate hyperplanes.

Expressing R_k in terms of covariant coordinates, we find

$$2.9 \qquad x'_j = \sum a_{jl}\bar{x}^{l\prime} = \sum_{l\neq k} a_{jl}\bar{x}^l + a_{jk}\left(\bar{x}^k - \frac{2x_k}{a_{kk}}\right)$$
$$= x_j - \frac{2a_{jk}x_k}{a_{kk}}$$

(14, p. 182**)**. Thus the matrix for the covariant transformation is the transpose of the matrix for the contravariant transformation (as it clearly must be for any transformation of period 2).

3. Groups in unitary 3-space. In the case of the ternary Hermitian form

$$3.1 \quad x^1\bar{x}^1 + x^2\bar{x}^2 + x^3\bar{x}^3 - \tfrac{1}{2}(x^2\bar{x}^3 + x^3\bar{x}^2 + x^3\bar{x}^1 + x^1\bar{x}^3 + cx^1\bar{x}^2 + \bar{c}x^2\bar{x}^1)$$

(which is semidefinite when $c = 1$), we have

$$a_{kk} = 1, \qquad a_{23} = a_{31} = -\tfrac{1}{2}, \qquad a_{12} = -\tfrac{1}{2}c.$$

The three reflections, expressed as matrices, are

$$3.2 \quad R_1 = \begin{pmatrix} -1 & \bar{c} & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 \\ c & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}.$$

We may easily verify that these three transformations R_k are of period 2, while their products R_2R_3 and R_3R_1 are of period 3. Certain abstract definitions obtained by Shephard and Todd **(31**, p. 299**)** suggest that we may have a sufficient set of defining relations for the abstract group $\{R_1, R_2, R_3\}$ as soon as we have specified also the periods of the products R_1R_2 and $R_1R_2R_3R_2$. Since

$$R_1R_2R_3R_2 = R_1R_3R_2R_3 \qquad \text{and} \qquad R_2R_3R_1R_3 = R_2R_1R_3R_1,$$

the relation $(R_1R_2R_3R_2)^m = E$ can be replaced by $(R_2R_3R_1R_2)^m = E$ or by $(R_3R_1R_2R_1)^m = E$.

To find how the periods of R_1R_2 and $R_1R_2R_3R_2$ depend on c, we observe that the characteristic equations for

$$R_1R_2 = \begin{pmatrix} c\bar{c} - 1 & -\bar{c} & \bar{c} + 1 \\ c & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_1R_2R_3R_2 = \begin{pmatrix} c\bar{c} + c + \bar{c} & -1 & -\bar{c} \\ c + 1 & 0 & -1 \\ c + 1 & -1 & 0 \end{pmatrix}$$

are respectively

$$(\lambda - 1)\{(\lambda + 1)^2 - c\bar{c}\lambda\} = 0, \quad (\lambda - 1)\{(\lambda + 1)^2 - (c + 1)(\bar{c} + 1)\lambda\} = 0.$$

Comparing these with the characteristic equation

$$(\lambda + 1)^2 - \left(2\cos\frac{\pi}{p}\right)^2\lambda = 0$$

for an ordinary rotation through $2\pi/p$, we see that the reflections 3.2 satisfy the relations

3.3 $$R_k^2 = (R_2R_3)^2 = (R_3R_1)^2 = (R_1R_2)^l = (R_1R_2R_3R_2)^m = E$$

if

3.4 $$c\bar{c} = 4\cos^2\frac{\pi}{l}, \quad (c+1)(\bar{c}+1) = 4\cos^2\frac{\pi}{m},$$

i.e., if $c = 2\cos\pi/l \cdot e^{\pi i/s}$, where s is given by

3.5 $$2\cos\frac{\pi}{s} = \left(4\cos^2\frac{\pi}{m} - 4\cos^2\frac{\pi}{l} - 1\right) \Big/ \left(2\cos\frac{\pi}{l}\right).$$

The most significant cases are worked out in Table I on page 269, where we use the abbreviations

$$\tau = \frac{\sqrt{5}+1}{2}, \quad \omega = \frac{-1+i\sqrt{3}}{2}$$

(so that $\tau^2 - \tau - 1 = 0$, $\omega^2 + \omega + 1 = 0$).

The form 3.1, having determinant

3.6 $$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \tfrac{1}{5}\begin{vmatrix} 2 & -c & -1 \\ -\bar{c} & 2 & -1 \\ -1 & -1 & 2 \end{vmatrix}$$
$$= \tfrac{1}{5}\{5 - c\bar{c} - (c+1)(\bar{c}+1)\}$$
$$= \tfrac{1}{5}\left(5 - 4\cos^2\frac{\pi}{l} - 4\cos^2\frac{\pi}{m}\right),$$

is positive definite if

$$4\cos^2\frac{\pi}{l} + 4\cos^2\frac{\pi}{m} < 5,$$

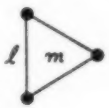
i.e., if

3.7 $$2\cos\frac{2\pi}{l} + 2\cos\frac{2\pi}{m} < 1.$$

When l (or m) has the value 3, m (or l) is unrestricted. The only other possibility is that l (or m) is 4 while m (or l) is 4 or 5. In each case, the results of Shephard and Todd will enable us to establish the sufficiency of 3.3 for an abstract definition of the group.

In the graphical notation of §1, the abstract group 3.3 is

3.8



for which a convenient abbreviation is $[1\ 1\ 1^l]^m$, with the l omitted when $l = 3$. (We place the first node at the top, the second at the bottom, and the third on the right.)

The substitution $R_2 \leftrightarrow R_2R_3R_2 = R_3R_2R_3$ shows that l and m are interchangeable:

3.9 $$[1\ 1\ 1^l]^m \simeq [1\ 1\ 1^m]^l.$$

In the case of $[1\ 1\ 1^m]^2$, we can replace $(R_1R_2R_3R_2)^3 = E$ or $(R_2R_3R_1R_3)^3 = E$ by the equivalent relation

$$(R_3R_1R_2)^2 = (R_1R_2R_3)^2,$$

which enables us to identify this group with Shephard's $[1\ 1;\ 1]^m$ or Todd's $G(m, m, 3)$ **(29**, p. 374; **31**, p. 277**)**. Thus Shephard's

is our

or

$[1, 1; 1]^m$ $\quad [1\ 1\ 1^m]^3 \quad \simeq \quad [1\ 1\ 1]^m,$

which is 3.3 with $l = 3$. It follows that the order of $[1\ 1\ 1]^m$ is $6m^2$. The first two values of m yield symmetric groups:

$$[1\ 1\ 1]^1 \simeq \mathfrak{S}_3, \quad \text{generated by } (2\ 3),\ (3\ 1),\ (1\ 2);$$

$$[1\ 1\ 1]^2 \simeq \mathfrak{S}_4, \quad \text{generated by } (1\ 4),\ (2\ 4),\ (3\ 4).$$

Since the relations 3.3 with $l = 1$ or 2 imply $m = 3$, there are no other groups with l or $m = 1$ or 2.

The substitution $R_2 \leftrightarrow R_1R_2R_1$ enables us to express $[1\ 1\ 1^4]^4$ in the form

$$R_2^2 = (R_2R_1R_2R_1)^2 = (R_2R_1)^3 = (R_1R_2)^4 = (R_2R_3)^4 = E,$$

which is No. 24, of order 336, in the list of Shephard and Todd **(31**, p. 299**)**. Similarly $[1\ 1\ 1^5]^4$ is No. 27, of order 2160. Thus the relations 3.3 suffice for an abstract definition in every case.

4. Extension to higher spaces. In accordance with the graphical symbolism described in §1, we derive from $[1\ 1\ 1^l]^m$ a group

$$[p\ q\ r^l]^m \simeq [q\ p\ r^l]^m$$

in unitary $(p + q + r)$-space, by adding "tails" of $p - 1$, $q - 1$, $r - 1$ branches to the three nodes in 3.8.

The isomorphism 3.9 is easily seen to be maintained when we add a tail to either or both of the first two nodes (which are joined by the marked branch):

4.1 $$[p\ q\ 1^l]^m \simeq [p\ q\ 1^m]^l.$$

When $l = 3$, we have the group

$$[p\ q\ r]^m,$$

which involves the three numbers p, q, r symmetrically.

As this notation scarcely differs from Shephard's (29, p. 371), we can make use of his results in order to interpret the graphical symbol both abstractly and geometrically. For instance, the group

$$[1\ 1\ 2^l]^3,$$

defined by

4.2 $$R_k^2 = (R_2R_3)^3 = (R_3R_1)^3 = (R_1R_2)^l = (R_1R_2R_3R_2)^3$$
$$= (R_1R_4)^2 = (R_2R_4)^2 = (R_3R_4)^3 = E,$$

is generated by reflections in the covariant coordinate hyperplanes when the metric is determined by the form 3.1 with the extra terms

$$x^4\bar{x}^4 - \tfrac{1}{2}(x^3\bar{x}^4 + x^4\bar{x}^3).$$

In Shephard's notation (29, p. 379), this group of order $24l^3$ is

$$[1\ 1; 2]^l.$$

Extending the tail, we have $[1\ 1\ (n-2)^l]^3$, which is his $[1\ 1; n-2]^l$, of order $l^{n-1}\,n!$. In fact, all his graphical symbols (29, pp. 371, 374, 379, 382, 383) can be amended by the following simple modification. Whenever he draws a triangle with a number (m or 4) inside, this inner number should be changed to 3.

Let $2^{-p-q-r}\,f(p, q, r)$ denote the determinant of the form corresponding to $[p\ q\ r^l]^m$; for instance,

$$f(1,1,2) = \begin{vmatrix} 2 & -c & -1 & 0 \\ -\bar{c} & 2 & -1 & 0 \\ -1 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} = 6 - 4\cos^2\frac{\pi}{l} - 8\cos^2\frac{\pi}{m}.$$

We easily find (cf. **17**, p. 426) the recursion formulae

$$\begin{aligned} f(p, q, r) &= 2f(p-1, q, r) - f(p-2, q, r) \\ &= f(q, p, r) \\ &= 2f(p, q, r-1) - f(p, q, r-2), \end{aligned}$$

with the initial values

$$f(1,1,1) = 5 - 4\cos^2\frac{\pi}{l} - 4\cos^2\frac{\pi}{m},$$

$$f(1,1,0) = 4\sin^2\frac{\pi}{l}, \quad f(p,0,r) = p + r + 1,$$

whence

$$f(p, q, r) = (p+1)(q+1) + r - 4pq(\cos^2\frac{\pi}{l} + r\cos^2\frac{\pi}{m})$$

(cf. **29**, p. 372). The necessary condition

$$f(p, q, r) > 0$$

yields the finite groups listed in Table II. The identification with Shephard's list can be completed as follows.

Setting $l = 3$, we see that $[p\ q\ r]^m$ occurs whenever

4.3 $$p + q + r + 1 - 4pqr\cos^2\frac{\pi}{m} > 0.$$

In particular, we have $[p\ q\ r]^2$ for all values of p, q, r; but this is merely an unusual way of generating $\mathfrak{S}_{p+q+r+1}$, the symmetry group of the regular simplex α_{p+q+r} in real $(p+q+r)$-space. For instance,

$[2\ 1\ 3]^2$

is $\mathfrak{S}_7$, generated by the transpositions

$$(1\ 5), \quad (1\ 4), \quad (3\ 4), \quad (3\ 6), \quad (6\ 7).$$
$$(2\ 4),$$

The criterion 4.3 shows that, when $m > 2$, the numbers p, q, r cannot all be greater than 1. Thus every such finite group (with $l = 3$) is expressible as

4.4 $$[p\ q\ 1]^m \simeq [p\ q\ 1^m]^2,$$

which we shall usually write with the 1 in the middle: $[p\ 1\ q]^m$. It is evidently Shephard's $[p\ 1; q]^m$ (**29**, p. 373).

The isomorphism 4.4, which is a special case of 4.1, has an interesting counterpart for $[1\ 1\ r^l]^3$. Consider, for instance, $[1\ 1\ 2^l]^3$, defined by 4.2. Instead of R_1, we may introduce a new generator

$$\overline{R}_1 = R_4R_3R_1R_3R_4$$

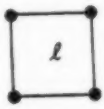
by writing

$$R_1 = R_3R_4\overline{R}_1R_4R_3.$$

The consequent relations

$$\begin{aligned}\overline{R}_1^2 = R_2^2 = R_3^2 = R_4^2 = (R_2R_3)^3 = (R_4\overline{R}_1)^3 \\ = (R_3R_2R_4\overline{R}_1R_4R_3)^l = (\overline{R}_1R_2)^3 \\ = (\overline{R}_1R_3)^2 = (R_2R_4)^2 = (R_3R_4)^3 = E\end{aligned}$$

are naturally represented by the graphical symbol



$[1\ 1\ 1\ 1^l]$.

More generally, an alternative symbol for $[1\ 1\ r^l]^2$ $(r = n - 2)$ consists of an n-gon with the mark l inside to indicate any one of n equivalent relations such as

$$(\overline{R}_1R_2R_3 \ldots R_nR_{n-1} \ldots R_2)^l = E.$$

When $n = 3$, we simply have $[1\ 1\ 1]^l$ as an alternative symbol for $[1\ 1\ 1^l]^2$.

5. Commutator subgroups. If the words in the defining relations for a given abstract group $\mathfrak{G}$ are such that each involves an even number of letters, then every element is either "even" or "odd" according to the parity of the number of letters occurring in any expression for it. The "even" elements form a subgroup $\mathfrak{G}^+$ of index 2.

When $\mathfrak{G}$ is $[p\ q\ r^l]^m$, one possible set of $p + q + r - 1$ generators for $\mathfrak{G}^+$ is provided by the products R_jR_k of period 3, represented in the graph by the unmarked branches. (We may ignore the branch marked l, even when $l = 3$.) Any such product is a commutator:

$$R_jR_k = R_kR_jR_kR_j = R_k^{-1}R_j^{-1}R_kR_j.$$

Moreover, any commutator

$$(R_i \ldots R_h)(R_k \ldots R_j)(R_h \ldots R_i)(R_j \ldots R_k)$$

is the product of an even number of R's. Hence

The commutator subgroup of $[p\ q\ r^l]^m$ *is* $[p\ q\ r^l]^{m+}$.

In particular, the commutator subgroup of $[1\ 1\ 1^l]^m$ is $[1\ 1\ 1^l]^{m+}$, whose generators

$$S_1 = R_1R_3, \quad S_2 = R_2R_3$$

satisfy the relations

5.1 $$S_1^3 = S_2^3 = (S_1S_2)^l = (S_1^{-1}S_2)^m = E.$$

These relations suffice for an abstract definition, since we can reconstruct 3.3 by adjoining to 5.1 a new element T, of period 2, which transforms both the S's into their inverses (and then defining $R_1 = S_1T$, $R_2 = TS_2$, $R_3 = T$). Thus

5.2 $$[1\ 1\ 1^l]^{m+} \simeq (3, 3 \mid l, m)$$

in the notation of **(12**, pp. 77–85**)**. (In order to interchange l and m, we merely have to replace one of the S's by its inverse.)

Setting $l = 3$, we recognize $[1\ 1\ 1]^{m+}$ as a group of order $3m^3$ considered by Edington **(21**, p. 208**)**. This is generated by the permutations

$$S_1 = (3m\ 1\ 2)(3\ 4\ 5)\ldots(3m-3\ \ 3m-2\ \ 3m-1),$$
$$S_2 = (2\ 3\ 4)(5\ 6\ 7)\ldots(3m-1\ \ 3m\ \ 1)$$

which yields, for $[1\ 1\ 1]^m$,

$$R_1 = (1\ 2)(4\ 5)\ldots(3m-2\ \ 3m-1),$$
$$R_2 = (2\ 3)(5\ 6)\ldots(3m-1\ \ 3m),$$
$$R_3 = (3\ 4)(6\ 7)\ldots(3m\ \ \ 1).$$

A more interesting case is

5.3 $$[1\ 1\ 1^4]^{4+} \simeq (3, 3 \mid 4, 4) \simeq LF(2, 7)$$

(12, pp. 83–84**)**. The generators

$$S_1 = (0\ 1\ \infty)(2\ 6\ 4) = \left(\frac{1}{1-x}\right) \pmod 7$$

$$S_2 = (0\ 2\ 3)(1\ 6\ 5) = (4x+2) \pmod 7$$

are transformed into their inverses by

$$(0\ 6)(1\ 2)(3\ 5)(4\ \infty) = \left(\frac{x+1}{2x-1}\right) \pmod 7.$$

Hence in this case we have an *inner* automorphism, and

5.4 $$[1\ 1\ 1^4]^4 \simeq \mathfrak{C}_2 \times LF(2, 7).$$

6. Exponents and invariants. By 2.9 with $a_{kk} = 1$, the reflection R_k (in covariant coordinates) is

$$x'_j = x_j - 2a_{jk}x_k \qquad (j = 1, \ldots, n)$$

or

$$x = x' - 2a_{jk}x'_k.$$

Since this is the same expression as for real reflections **(16**, pp. 766–767**)**, the characteristic equation for the product

$$R_1 R_2 \dots R_n$$

is again

$$6.1 \qquad \begin{vmatrix} \frac{1}{2}(\lambda+1) & a_{12}\lambda & a_{13}\lambda & \dots & a_{1n}\lambda \\ a_{21} & \frac{1}{2}(\lambda+1) & a_{23} & \dots & a_{2n}\lambda \\ \dots & & & \dots & \\ a_{n1} & a_{n2} & a_{n3} & & \frac{1}{2}(\lambda+1) \end{vmatrix} = 0$$

(25, p. 20**)**. In the case of $[1\ 1\ 1^{l}]^{m}$, given by 3.1, we take $a_{12} = -\frac{1}{2}c$, $a_{21} = -\frac{1}{2}\bar{c}$, while the remaining a_{jk} $(j \neq k)$ are all $-\frac{1}{2}$, so that the equation is

$$6.2 \qquad \begin{vmatrix} \lambda+1 & -c\lambda & -\lambda \\ -\bar{c} & \lambda+1 & -\lambda \\ -1 & -1 & \lambda+1 \end{vmatrix} = 0$$

or

$$\lambda^3 - (c\bar{c} + c - 1)\lambda^2 - (c\bar{c} + \bar{c} - 1)\lambda + 1 = 0.$$

This is unchanged when c is replaced by $-\bar{c} - 1$; therefore, by 3.4, it involves l and m symmetrically.

In the case of $[1\ 1\ 1]^{m}$, we have $c = e^{2\pi i/m}$, the equation reduces to

$$\lambda^3 - c\lambda^2 - \bar{c}\lambda + 1 = 0$$

(31, p. 295, with c instead of θ^{-1}**)**, and the roots are

$$c,\ \pm\bar{c}^{\frac{1}{2}}.$$

In the case of $[1\ 1\ 1^{l}]^{4}$, we have $(c+1)(\bar{c}+1) = 2$, and the equation reduces to

$$\lambda^3 + \bar{c}\lambda^2 + c\lambda + 1 = 0$$

with $c = \frac{1}{2}(-1 + i\sqrt{7})$ when $l = 4$, and $c = \omega\tau$ when $l = 5$ **(31**, p. 296**)**.

Similarly, we can reconstruct the characteristic equation for $R_1 R_2 \dots R_n$ when $n > 3$:

for $[1\ 1\ 2]^{4}$, $\quad \lambda^4 - i\lambda^3 - \lambda^2 + i\lambda + 1 = 0$;
for $[2\ 1\ 2]^{3}$, $\quad \lambda^5 - \omega\lambda^4 + \bar{\omega}\lambda^3 + \omega\lambda^2 - \bar{\omega}\lambda + 1 = 0$;
for $[2\ 1\ 3]^{6}$, $\quad \lambda^6 - \omega\lambda^5 + \bar{\omega}\lambda^4 - \lambda^3 + \omega\lambda^2 - \bar{\omega}\lambda + 1 = 0$

(31, p. 298**)**. The last equation arises in the form

$$\begin{vmatrix} \lambda+1 & -\lambda & 0 & 0 & 0 & 0 \\ -1 & \lambda+1 & -\omega\lambda & -\lambda & 0 & 0 \\ 0 & -\bar{\omega} & \lambda+1 & -\lambda & 0 & 0 \\ 0 & -1 & -1 & \lambda+1 & -\lambda & 0 \\ 0 & 0 & 0 & -1 & \lambda+1 & -\lambda \\ 0 & 0 & 0 & 0 & -1 & \lambda+1 \end{vmatrix} = 0,$$

which is the natural extension of 6.2 (with $c = \omega$) from $[1\ 1\ 1]^3$ to $[2\ 1\ 3]^3$. If the equation for $[p\ q\ r^{\dagger}]^m$ is $f_r(\lambda) = 0$, we can use the recursion formula

$$f_r(\lambda) = (\lambda+1)f_{r-1}(\lambda) - \lambda f_{r-2}(\lambda).$$

By taking p, q, r in such an order that $q = 1$, we may name the n generators (as in the above example) in the order of a Hamiltonian path **(14**, p. 8**)**. We start at the tip of the first tail, pass to the second node (where there is no second tail), and proceed along the third tail.

In the case of $[1\ 1\ (n-2)^{\dagger}]^3$, we have $c = -\bar{b} - 1$ where $b = e^{2\pi i/l}$, the equation is

$$(\lambda - b)\ (\lambda^{n-1} - \bar{b}) = 0$$

(31, p. 295, with b instead of θ^{-1}**)**, and the roots are

$$\epsilon^{n-1},\ \epsilon^{l-1},\ \epsilon^{2l-1},\ \ldots,\ \epsilon^{(n-1)l-1},$$

where $\epsilon = e^{2\pi i/(n-1)l}$.

In each case, to compute h, the period of $R_1R_2 \ldots R_n$, we observe that the characteristic roots are powers

$$\epsilon^{m_j} \qquad (j = 1, 2, \ldots, n; \quad m_1 \leqslant m_2 \leqslant \ldots \leqslant m_n)$$

of a primitive hth root of unity, $\epsilon = e^{2\pi i/h}$. Our choice of the positive value of s in 3.5, combined with the "natural" order of $R_1, R_2, \ldots, R_n$ in the chosen product, has the effect that

6.3 $$m_n = h - 1.$$

(By reversing either of these conventions we would have had $m_1 = 1$ instead of 6.3. In the case of real reflections, this important distinction disappears, as we have both $m_1 = 1$ and $m_n = h - 1$.) The actual values are given in Table II on page 270 **(**cf. **16**, p. 771**)**. It was proved by Shephard and Todd **(31**, p. 289**)** that any finite group of unitary transformations which possesses a basic set of n invariant polynomial forms is a group generated by unitary reflections. Moreover, when the n invariant forms are chosen so as to have the smallest possible degrees, the product of the degrees is equal to the order of the group. They observed **(31**, pp. 283, 294**)** that these degrees are just the numbers

$$m_j + 1 \qquad (j = 1, 2, \ldots, n).$$

Therefore the Jacobian J of the n forms has degree $\sum m_j$, which is equal to the number of reflections in the group **(31**, p. 290**)**. Since J changes sign when operated on by a reflection of period 2, it follows that J factorizes into $\sum m_j$ linear forms which, when equated to zero, give the $\sum m_j$ reflecting hyperplanes **(30**, p. 47**)**.

For instance, the group $[1\ 1\ 1^4]^4$ has exponents $m_1 = 3$, $m_2 = 5$, $m_3 = 13$. Klein **(26**, pp. 446–448**)** described invariants f, ∇, C, of degrees 4, 6, 14, and observed that their Jacobian factorizes into 21 linear forms which, when

equated to zero, give the axes of 21 harmonic homologies transforming the plane quartic curve $f = 0$ into itself.

Incidentally, we have the simple expression

$$(m_1 + 1)\,(m_2 + 1) \ldots (m_n + 1) \tag{6.4}$$

for the order of the group.

7. Central quotient groups. Another interesting property of the degrees $m_j + 1$ is that their greatest common divisor

$$k = (m_1 + 1, \ \ldots \ , m_n + 1)$$

is equal to the order of the centre of the group. In fact, since

$$m_j \equiv -1 \pmod{k},$$

the characteristic roots of the element

$$(R_1 \ldots R_n)^{h/k}$$

are

$$\epsilon^{m_j h/k} = \epsilon^{2\pi i m_j/k} = \epsilon^{-2\pi i/k},$$

i.e., they are all equal, which means that this is the central transformation $\epsilon^{-2\pi i/k} I$, of period k **(31**, p. 280**)**.

By regarding the coordinates x_j or x^j as being homogeneous, we pass from complex affine n-space to complex projective $(n - 1)$-space. Instead of a group $\mathfrak{G}$ generated by involutory reflections, we now have the central quotient group

$$\mathfrak{G}/\mathfrak{C}_k,$$

generated by harmonic homologies **(31**, p. 275**)**. Its abstract definition is given by the graphical symbol along with the single extra relation

$$(R_1 \ldots R_n)^{h/k} = E. \tag{7.1}$$

In the complex projective plane, we have two groups:

$$[1\ 1\ 1^4]^4/\mathfrak{C}_2 \simeq LF(2, 7),$$

defined by 3.3 with $l = m = 4$ and $(R_1R_2R_3)^7 = E$ (cf. 5.3, 5.4), which is Klein's collineation group of order 168, containing 21 harmonic homologies **(26**, p. 440**)**; and

$$[1\ 1\ 1^5]^4/\mathfrak{C}_6 \simeq \mathfrak{A}_6,$$

defined by 3.3, with $l = 5$, $m = 4$ and $(R_1R_2R_3)^5 = E$, which is Valentiner's collineation group **(35**, p. 227**)**, of order 360, containing 45 harmonic homologies. When the latter is represented as the alternating group of degree six **(37)**, the homologies appear as the double transpositions; e.g., the generators may be taken to be

$$R_1 = (1\ 3)\ (4\ 6), \quad R_2 = (3\ 4)\ (5\ 6), \quad R_3 = (2\ 3)\ (5\ 6).$$

In complex projective 3-space, we have

$$[1\ 1\ 2]^4/\mathfrak{C}_4,$$

defined by

$$R_j^2 = (R_3R_5)^3 = (R_3R_1)^3 = (R_1R_2)^3 = (R_1R_2R_3R_2)^4$$
$$= (R_1R_4)^2 = (R_2R_4)^2 = (R_3R_4)^3 = (R_1R_2R_3R_4)^5 = E,$$

which is Bagnera's collineation group (**1**, p. 33) of order 1920, containing 40 harmonic homologies. This leaves invariant the Maschke quartic surface

$$x^4 + y^4 + z^4 + t^4 - 12xyzt = 0$$

(**27**, p. 504; **20a**), and a configuration consisting of five tetrahedra (**29**, p. 382), namely the tetrahedra (ab), (ac), (ad), (ae), (af) of Hudson (**24**, p. 43).

In complex projective 4-space, we have

$$[2\ 1\ 2]^3/\mathfrak{C}_2,$$

which is Mitchell's collineation group of order $36 \cdot 6!$, containing 45 harmonic homologies whose centres are the nodes of the Burkhardt primal (**2**; **33**; **34**). Since the word

$$(R_1R_2R_3R_4R_5)^3$$

involves an odd number of letters, this element, which occurs in $[2\ 1\ 2]^3$ as the "central inversion", does not belong to the commutator subgroup $[2\ 1\ 2]^{3+}$; hence the central quotient group

$$[2\ 1\ 2]^3/\mathfrak{C}_2 \simeq [2\ 1\ 2]^{3+}$$

is the simple group of order 25920, and $[2\ 1\ 2]^3$ is its direct product with $\mathfrak{C}_2$.

In complex projective 5-space, we have

$$[2\ 1\ 3]^3/\mathfrak{C}_6$$

(**29**, p. 375), which is Mitchell's collineation group of order $18 \cdot 9!$, containing 126 harmonic homologies whose centres form the Mitchell-Hamill configuration (**23**, p. 402). Its commutator subgroup is the simple group of order $9 \cdot 9!$:

$$[2\ 1\ 3]^{3+}/\mathfrak{C}_6 \simeq HO(4, 3^2) \simeq PU_4^+(\mathbf{F}_9)$$

(**23**, p. 451; **19**, p. 310; **20**, p. 48).

8. Infinite groups. When the form 2.1 is not definite but only semi-definite, we have an infinite group generated by reflections 2.9 in n hyperplanes $x_k = 0$ forming an $(n-1)$-dimensional simplex. In fact, since $\det(a_{jk}) = 0$, there exist constants $z^1, \ldots, z^n$ such that

$$\sum z^j a_{jk} = 0$$

and therefore, by 2.9,

$$\sum z^j x_j' = \sum z^j x_j.$$

Thus we may regard the group as operating in the $(n-1)$-space

$$\sum z^j x_j = 1,$$

in which the semidefinite form determines a unitary metric, and the R_k's appear as reflections in the simplex formed by the sections of the coordinate hyperplanes.

Adapting 3.7, we see that $[1\ 1\ 1^l]^m$ is infinite when

$$2\cos\frac{2\pi}{l} + 2\cos\frac{2\pi}{m} = 1.$$

When $l = 3$ and $m = \infty$, this is the group

$$R_k^2 = (R_2R_3)^3 = (R_3R_1)^3 = (R_1R_2)^3 = E$$

generated by real reflections in the sides of an ordinary equilateral triangle **(14**, p. 78**)**. The only other two-dimensional instance is

$$[1\ 1\ 1^4]^6 \simeq [1\ 1\ 1^6]^4,$$

whose commutator subgroup $(3,\ 3\ |\ 4,\ 6)$ is already known to be infinite **(12**, p. 95; **13**, p. 250**)**.

The group $[1\ 1\ 1^l]^3$, of order $6l^3$, defined by 3.3 with $m = 3$, has two infinite extensions

given by the extra relations

$$R_0^2 = (R_0R_1)^3 = (R_0R_2)^3 = (R_0R_1R_2R_1)^3 = E,$$
$$R_4^2 = (R_1R_4)^3 = (R_2R_4)^3 = (R_3R_4)^4 = E,$$

respectively. When $l = 2$, they reduce to the real groups

$$\square \text{ and } \begin{bmatrix} 3, & 3 \\ & 4 \end{bmatrix}$$

(14, pp. 84, 85**)**. More generally, $[1\ 1\ r^l]^3$ is a subgroup of the infinite group

(but on p. 265 we shall see that, as a geometrical group, this is discrete only if $l = 2, 3, 4$ or 6).

Similarly, the finite groups

$$[1\ 1\ 1^4]^4, \quad [2\ 1\ 1^4]^3, \quad [1\ 1\ 2]^4, \quad [2\ 1\ 3]^3$$

have infinite extensions

$$[1\ 1\ 2^4]^4, \quad [2\ 1\ 2^4]^3, \quad [1\ 1\ 3]^4, \quad [2\ 1\ 4]^3.$$

The last three were discovered by Shephard (**29**, pp. 382, 383, 381). But we must remember to change the mark inside his triangles from 4 to 3. His scheme does not include our

$$[1\ 1\ 2^4]^4,$$

whose determinant is

$$\tfrac{1}{16}\begin{vmatrix} 2 & -c & -1 & 0 \\ -\bar{c} & 2 & -1 & 0 \\ -1 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} = \tfrac{1}{16}\{6 - c\bar{c} - 2(c+1)(\bar{c}+1)\} = 0$$

(since $c\bar{c} = (c+1)(\bar{c}+1) = 2$).

9. Orthogonal coordinates. By working out the cofactors in the determinant 3.6 for $[1\ 1\ 1^l]^m$, we find the adjoint form of 3.1 to be a numerical multiple of

9.1
$$\begin{aligned} &3(x_1\bar{x}_1 + x_2\bar{x}_2) + (4 - c\bar{c})x_3\bar{x}_3 \\ &+ (c+2)(x_2\bar{x}_3 + x_3\bar{x}_1) + (\bar{c}+2)(x_3\bar{x}_2 + x_1\bar{x}_3) \\ &+ (2\bar{c}+1)x_1\bar{x}_2 + (2c+1)x_2\bar{x}_3. \end{aligned}$$

(See Table I, on page 269, for the values of c.)

In the case of $[1\ 1\ 1]^m$, where $l = 3$ and

$$c = e^{2\pi i/m},$$

9.1 may be expressed as

$$\begin{aligned} &(x_1 + cx_2 + x_3)(\bar{x}_1 + \bar{c}\bar{x}_2 + \bar{x}_3) + (\bar{c}x_1 + x_2 + x_3)(c\bar{x}_1 + \bar{x}_2 + \bar{x}_3) \\ &+ (x_1 + x_2 + x_3)(\bar{x}_1 + \bar{x}_2 + \bar{x}_3). \end{aligned}$$

In terms of orthogonal coordinates

$$\xi_1 = x_1 + cx_2 + x_3, \quad \xi_2 = \bar{c}x_1 + x_2 + x_3, \quad \xi_3 = x_1 + x_2 + x_3,$$

the form is $\xi_1\bar{\xi}_1 + \xi_2\bar{\xi}_2 + \xi_3\bar{\xi}_3$, and the reflecting planes are $x_k = 0$ where

9.2 $(1-c)x_1 = c(\xi_2 - \xi_3)$, $(1-c)x_2 = \xi_3 - \xi_1$, $(1-c)x_3 = \xi_1 - c\xi_2$,

so that the reflections themselves are:

$$\begin{aligned} &R_1, \quad (\xi_1, \xi_2, \xi_3) \rightarrow (\xi_1, \xi_3, \xi_2);\\ &R_2, \quad (\xi_1, \xi_2, \xi_3) \rightarrow (\xi_3, \xi_2, \xi_1);\\ &R_3, \quad (\xi_1, \xi_2, \xi_3) \rightarrow (c\xi_2, \bar{c}\xi_1, \xi_3) \end{aligned}$$

(31, pp. 276, 295**)**.

In the case of $[1\ 1\ 1^l]^3$, we have

$$c = -\bar{b} - 1, \qquad b = e^{2\pi i/l},$$

and the form 9.1 is

$$\begin{aligned} &3(x_1\bar{x}_1 + x_2\bar{x}_2) + (b-1)(\bar{b}-1)x_3\bar{x}_3\\ &- (\bar{b}-1)(x_2\bar{x}_3 + x_3\bar{x}_1) - (b-1)(x_3\bar{x}_2 + x_1\bar{x}_3)\\ &\quad - (2b+1)x_1\bar{x}_2 - (2\bar{b}+1)x_2\bar{x}_1\\ &= (x_1 - x_2)(\bar{x}_1 - \bar{x}_2) + (bx_1 - x_2)(\bar{b}\bar{x}_1 - \bar{x}_2)\\ &\quad + \{bx_1 - x_2 + (b-1)x_3\}\{\bar{b}\bar{x}_1 - \bar{x}_2 + (\bar{b}-1)\bar{x}_3\}. \end{aligned}$$

In terms of orthogonal coordinates

$$\xi_1 = b(x_1 - x_2), \quad \xi_2 = bx_1 - x_2, \quad \xi_3 = bx_1 - x_2 + (b-1)x_3,$$

the reflecting planes are $x_k = 0$ where

9.3 $(1-b)x_1 = \bar{b}\xi_1 - \xi_2$, $(1-b)x_2 = \xi_1 - \xi_2$, $(1-b)x_3 = \xi_2 - \xi_3$,

so that the reflections themselves are:

$$\begin{aligned} &R_1, \quad (\xi_1, \xi_2, \xi_3) \rightarrow (b\xi_2, \bar{b}\xi_1, \xi_3);\\ &R_2, \quad (\xi_1, \xi_2, \xi_3) \rightarrow (\xi_2, \xi_1, \xi_3);\\ &R_3, \quad (\xi_1, \xi_2, \xi_3) \rightarrow (\xi_1, \xi_3, \xi_2). \end{aligned}$$

The resemblance of 9.2 and 9.3 illustrates the fact that $[1\ 1\ 1^l]^m$ and $[1\ 1\ 1^m]^l$ are different ways of generating the same group.

In the case of $[1\ 1\ 1^4]^4$, we have

9.4 $$c = \tfrac{1}{2}(-1 + i\sqrt{7}) = \beta + \beta^2 + \beta^4, \qquad \beta = e^{2\pi i/7},$$

so that c and $\bar{c}$ are the roots of the equation

9.5 $$x^2 + x + 2 = 0,$$

and the form 9.1 is

$$\begin{aligned} &3(x_1\bar{x}_1 + x_2\bar{x}_2) + 2x_3\bar{x}_3 - c^2(x_2\bar{x}_3 + x_3\bar{x}_1)\\ &\qquad - \bar{c}^2(x_3\bar{x}_2 + x_1\bar{x}_3) - (c - \bar{c})(x_1\bar{x}_2 - x_2\bar{x}_1)\\ &= \tfrac{1}{2}\{x_1\bar{x}_1 + (x_1 + cx_2)(\bar{x}_1 + \bar{c}\bar{x}_2)\\ &\qquad + (\bar{c}^2x_1 + c^2x_2 - 2x_3)(c^2\bar{x}_1 + \bar{c}^2\bar{x}_2 - 2\bar{x}_3)\}. \end{aligned}$$

In terms of orthogonal coordinates

$$\xi_1 = x_1, \quad \xi_2 = x_1 + cx_2, \quad \xi_3 = \bar{c}^2 x_1 + c^2 x_2 - 2x_3,$$

the reflecting planes are $x_k = 0$, where

$$9.6 \qquad x_1 = \xi_1, \quad x_2 = \tfrac{1}{2}\bar{c}(-\xi_1 + \xi_2), \quad x_3 = \tfrac{1}{2}(-\xi_1 + c\xi_2 - \xi_3),$$

in agreement with Shephard and Todd **(31**, p. 295**)**, whose α and x_j are our $-c$ and ξ_j. The reflections themselves are:

R_1, reversing the sign of ξ_1;

R_2, interchanging ξ_1 and ξ_3;

$$R_3, \begin{cases} \xi_1' = \tfrac{1}{2}(\xi_1 + c\xi_2 - \xi_3), \\ \xi_2' = \tfrac{1}{2}\bar{c}(\xi_1 + \xi_3), \\ \xi_3' = \tfrac{1}{2}(-\xi_1 + c\xi_2 + \xi_3). \end{cases}$$

In terms of the ξ's, the quartic invariant (Klein's f, mentioned near the end of p. 254) is

$$9.7 \qquad \xi_1^4 + \xi_2^4 + \xi_3^4 + 3\bar{c}(\xi_2^3\xi_3^3 + \xi_3^3\xi_1^3 + \xi_1^3\xi_2^3).$$

One can soon verify that this is transformed into itself not only by R_1 and R_2 but also by R_3 **(**cf. **22**, p. 338 with $b/a = 3\bar{c}$**)**.

The remaining groups may be treated similarly, but the details are omitted because the results have all been obtained another way by Shephard **(29**, p. 373**)** and Todd **(31**, pp. 296, 298**)**.

10. Polytopes and honeycombs. Let O denote the origin, i.e., the point of intersection of the hyperplanes of the generating reflections $R_1, \ldots, R_n$. Clearly, O is invariant, and the images of any other point are all equidistant from O. In particular, let P be a point on the line of intersection of $n-1$ of the n hyperplanes, say all except the kth. Then P is invariant under the subgroup generated by all the R's except R_k; but R_k transforms P into another point Q, and the whole group transforms P and Q into the vertices of a configuration called a *polytope* **(28**, p. 83**)** (or, when $n = 2$ or 3, a *polygon* or a *polyhedron*). Any subset of the R's (including R_k) generates a subgroup which transforms P into the vertices of a sub-configuration called an *element* of the polytope (e.g., a *side* of the polygon, or an *edge* or *face* of the polyhedron).

Similarly, when we have an infinite discrete group in $(n-1)$-space, generated by reflections $R_1, \ldots, R_n$ in n hyperplanes forming a simplex, as in §8, the images of the kth vertex P of the simplex are said to form a *honeycomb* **(29**, pp. 364, 375**)**. The kth reflection R_k transforms P into another vertex Q of the honeycomb, and the subgroup generated by the remaining $n-1$ R's transforms Q into the *vertex figure* at P.

When the group is symbolized by a graph, the polytope or honeycomb is indicated by drawing a ring round the kth node **(14**, p. 87; **29**, p. 375**)**, and an element can be found by deleting one or more nodes (along with any

branches that occur there). Adapting the notation of Coxeter **(9**, p. 331**)** and Shephard **(29**, p. 378**)**, we let

$$(p_p\ q\ r^l)^m, \quad (p\ q_q\ r^l)^m, \quad (p\ q\ r_r{}^l)^m$$

denote the polytopes (or honeycombs) indicated by ringing the tip of the first, second, or third tail. Of course,

$$(p_p\ q\ r^l)^m = (q\ p_p\ r^l)^m,$$

and there is complete symmetry when $l = 3$ (in which case the l is omitted).

The same reasoning that led to 3.9 and 4.1 shows that the two polytopes

$$(p_p\ q\ 1^l)^m, \quad (p_p\ q\ 1^m)^l$$

have the same vertices; in fact, they have the same j-dimensional elements for $j = 0, 1, \ldots, p$. Similarly, $(1_1\ 1\ r^l)^2$ has the same vertices and edges as

$$(1_1\ 1 \ldots 1)^l,$$

whose graphical symbol consists of an n-gon ($n = r + 2$) with one vertex ringed (see p. 251).

To compute the number of vertices of $(p_p\ q\ r^l)^m$ or of $(p\ q\ r_r{}^l)^m$, as in Table III, we divide the order of the group $[p\ q\ r^l]^m$ by the order of the subgroup that leaves one vertex invariant. This subgroup is given by reducing p to $p - 1$, or r to $r - 1$, respectively. To make this rule apply to $(p\ 1_1\ r^l)^m$ or to $(p\ q\ 1_1{}^l)^m$, we have to interpret the symbols

$$[p\ 0\ r^l]^m, \quad [p\ q\ 0^l]^m.$$

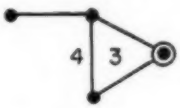
We see from the graph that $[p\ 0\ r^l]^m$ (including $[p\ 0\ r]^m$ or $[p\ r\ 0]^m$ as the special case when $l = 3$) is the symmetric group

$$[3, 3, \ldots, 3] \simeq \mathfrak{S}_{p+r+1},$$

represented by a simple chain of $p + r - 1$ unmarked branches. For instance, the six-dimensional polytope $(2\ 1_1\ 3)^2$ has

$$\frac{108 \cdot 9!}{6!} = 54432$$

vertices. Similarly, by removing the ringed node from



$$(2\ 1\ 1_1{}^4)^2,$$

we see that $[2\ 1\ 0^4]^2$ is the extended octahedral group $[3, 4]$, of order 48 **(14**, pp. 82, 85**)**.

The only remaining case of $[p\ q\ 0^l]^m$ with $l > 3$ is $[1\ 1\ 0^l]^m$, which is the dihedral group $[l]$, of order $2l$, defined by

$$R_1^2 = R_2^2 = (R_1R_2)^l = E.$$

This is the case of the polyhedron

10.1

$$(1\ 1\ 1_1{}^l)^m,$$

whose faces are equilateral triangles. The initial vertex P is taken on the line of intersection of the first two reflecting planes ("mirrors"). The third reflection transforms P into another vertex Q, such that PQ is an edge (symbolized by the ringed node alone). The first two reflections, whose product is of period l, transform PQ into $2l$ edges PQ, PQ', etc. (see Fig. 1). Hence this is a polyhedron of type $\{3, 2l\}$: the faces are triangles such as PQQ', and every vertex belongs to $2l$ of them. By 6.4, the order of the group $[1\ 1\ 1^l]^m$ is

$$g = (m_1 + 1)\ (m_2 + 1)\ (m_3 + 1),$$

where m_1, m_2, m_3 are computed from the roots $e^{2\pi i m_j/h}$ of the equation 6.2 (see Table II). Since there is one element of the group for every half-edge of $(1\ 1\ 1_1{}^l)^m$, this polyhedron of type $\{3, 2l\}$ has $\frac{1}{2}g$ edges, $\frac{1}{3}g$ triangular faces, and $g/2l$ vertices.

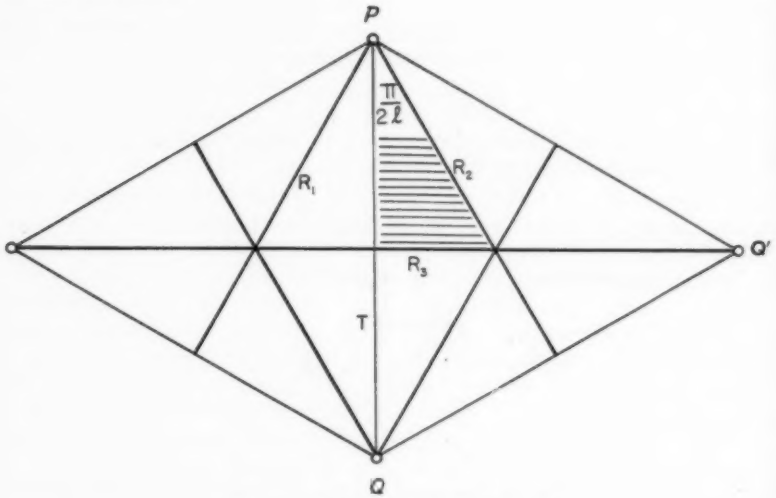


FIG. 1: A face PQQ' of $(1\ 1\ 1_1{}^l)^m$

When $l = 2$ (and therefore, by 3.3, $m = 3$), we have the octahedron $\{3, 4\}$. In other cases we may obtain a real interpretation by regarding the real and imaginary parts of the ξ's as Cartesian coordinates in a Euclidean space of twice as many dimensions. When so interpreted, $(1\ 1\ 1_l)^m$ appears as a regular *skew* polyhedron **(11)** in Euclidean 6-space. Shephard **(28)** did not include it in his list of regular complex polyhedra because, as we shall see in § 11, the symmetry operation interchanging two adjacent faces is not, in general, a unitary reflection.

When $l = 3$ we have $(1\ 1\ 1_l)^m$, and 9.2 shows that the first two mirrors meet along the line $\xi_1 = \xi_2 = \xi_3$, so we take P to be the point $(1, 1, 1)$. Applying the reflections, we find Q to be $(c, \bar{c}, 1)$, and Q' $(1, \bar{c}, c)$; altogether we obtain the m^2 points

$$(c^{k_1},\ c^{k_2},\ c^{k_3}), \qquad c = e^{2\pi i/m}, \qquad k_1 + k_2 + k_3 \equiv 0 \pmod{m}.$$

Topologically, $(1\ 1\ 1_l)^m$ is a triangulation of the torus, namely the regular map

$$\{3, 6\}_{m,0}$$

(15, p. 421**)**, which thus has a metrical realization as a skew polyhedron in Euclidean 6-space. Fig. 2 shows the case $m = 4$, with the values of $k_1k_2k_3$ marked at each vertex (so that P, Q, Q' are 000, 130, 031).

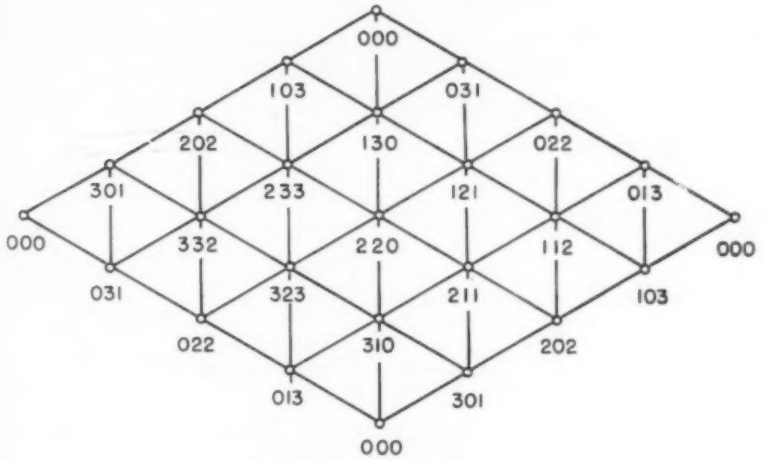


FIG. 2: $(1\ 1\ 1_l)^4 = \{3, 6\}_{4,0}$.

Similarly in n dimensions, reflections in the n hyperplanes

$$\xi_2 = \xi_3,\ \xi_3 = \xi_4,\ \ldots,\ \xi_n = \xi_1,\ \xi_1 = c\xi_2 \qquad (c = e^{2\pi i/m})$$

generate the group

$$[1\ 1\ \ldots\ 1]^m,$$

which transforms the point $(1, 1, \ldots, 1)$ into the m^{n-1} points

$$(c^{k_1}, c^{k_2}, \ldots, c^{k_n}), \qquad \sum k_j \equiv 0 \pmod{m}.$$

These were shown by Shephard **(29**, p. 378**)** to be the vertices of the "fractional γ polytope"

$$(1_1\ 1\ r^m)^3 = \tfrac{1}{m}\gamma_n^m, \qquad r = n - 2.$$

We now recognize the same m^{n-1} points as the vertices of the new polytope

$$(1_1\ 1\ \ldots\ 1)^m,$$

whose $(n-1)$-dimensional elements are the various truncations of the regular simplex α_{n-1}, just as they are in the limiting case of the real honeycomb

$$(1_1\ 1\ \ldots\ 1)^\infty = \alpha_{n-1}\mathrm{h}$$

(7, p. 366; **14**, p. 205, footnote**)**.

In the case of $(1\ 1\ 1_1{}^l)^3$, 9.3 shows that the first two mirrors meet along the line $\xi_1 = \xi_2 = 0$, so we take P to be $(0, 0, 1)$ and obtain the $3l$ points whose coordinates are the permutations of

$$(b^k, 0, 0), \qquad b = e^{2\pi i/l}, \qquad k = 0, 1, \ldots, l-1$$

(29, p. 377**)**. Accordingly, $(1\ 1\ 1_1{}^l)^3$ is the "generalized octahedron" β_3^l.

Similarly in n dimensions, reflections in the n hyperplanes

$$\xi_1 = b\xi_2, \quad \xi_1 = \xi_2, \quad \xi_2 = \xi_3, \quad \ldots, \quad \xi_{n-1} = \xi_n$$

generate the group

$$[1\ 1\ r^l]^3, \qquad r = n - 2;$$

and the nl points obtained by permuting

$$(b^k, 0, 0, \ldots, 0) \qquad (k = 0, 1, \ldots, l-1)$$

are the vertices of the generalized cross polytope

$$(1\ 1\ r_r^{l_r})^3 = \beta_n^l$$

(10, p. 287**)**. When $l = 2$, 3, 4 or 6, this polytope β_n^l is the vertex figure of the honeycomb

which is derived from the origin by applying the group $[1\ 1\ r^l]^3$ along with the $(n+1)$th reflection

$$(\xi_1, \ldots, \xi_{n-1}, \xi_n) \rightarrow (\xi_1, \ldots, \xi_{n-1}, 1 - \xi_n).$$

The reason for restricting l to the "crystallographic" values

$$2, 3, 4, 6$$

may already be seen when $n = 1$ (although the graphical symbol applies only when $n \geq 3$). The one-dimensional honeycomb consists of the points (ξ) whose single coordinate ξ is derived from 0 by the two operations of multiplying by b and subtracting from 1. In the Argand plane, these operations appear as rotation through $2\pi/l$ about the origin and reflection in the line $x = \frac{1}{2}$. The group so generated is not discrete when $l = 5$, nor when $l > 7$. (The proof, ascribed to Wigner in **(14**, p. 65**)**, is really due to Barlow **(3**, p. 17**)**.) When $l = 2$, the values of ξ are the real integers; when $l = 4$ they are the Gaussian integers $x + yi$; when $l = 6$ they are the Eisenstein integers $u + v\omega$; when $l = 3$ they are two-thirds of the Eisenstein integers, namely those for which $u + v \equiv 0$ or $1 \pmod 3$ **(14**, p. 64**)**. The corresponding honeycombs in the Argand plane are

$$\{\infty\}, \quad \{4, 4\}, \quad \{3, 6\}, \quad \{6, 3\}.$$

For greater values of n, we have points $(\xi_1, \ldots, \xi_n)$ where each of the n coordinates is restricted to the appropriate one of the four classes. The corresponding real honeycombs are the "rectangular products"

$$\{\infty\}^n = \delta_{n+1}, \quad \{4, 4\}^n = \delta_{2n+1}, \quad \{3, 6\}^n, \quad \{6, 3\}^n$$

(7, pp. 353–354; **14**, pp. 123–124**)**.

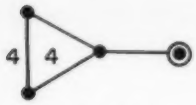
Returning to unitary 3-space, let us investigate the polyhedron

$$(1\ 1\ 1_1^4)^4,$$

for which 9.6 yields the line $\xi_1 = \xi_2 = 0$ again. To avoid fractions, we now take P to be $(0, 0, 2)$, whose images are the $6 + 12 + 24 = 42$ permutations of

$$(0, 0, \pm 2), \quad (\pm c, \pm c, 0), \quad (\pm 1, \pm 1, \pm \bar{c}),$$

where c is given by 9.4. Since each of these 42 vertices belongs to 8 edges, there are altogether 168 edges and 112 triangular faces. This polyhedron $(1\ 1\ 1_1^4)^4$ is the vertex figure of the honeycomb



$(1\ 1\ 2_2^4)^4$,

whose group $[1\ 1\ 2^4]^4$ is derived from $[1\ 1\ 1^4]^4$ by adjoining the fourth reflection

$$\xi_1' = \xi_1, \quad \xi_2' = \xi_2, \quad \xi_3' = 2 - \xi_3.$$

Applying this group to the origin, we obtain the points whose coordinates ξ_1, ξ_2, ξ_3 satisfy the congruences

10.2 $$\xi_1 \equiv \xi_2 \equiv \xi_3 \pmod{c}, \quad \xi_1 + \xi_2 + \xi_3 \equiv 0 \pmod{\bar{c}}$$

in the domain of algebraic integers generated by the roots (c and $\bar{c}$) of the equation 9.5. From the nature of this prescription, we see that the vertices of $(1\ 1\ 2_2^4)^4$, like those of $(2\ 1\ 4_4)^2$ **(18**, p. 386**)**, form a *lattice*.

The polyhedron $(1_1\ 1\ 1^4)^4$ (see Table III) is less interesting than $(1\ 1\ 1_1^4)^4$ because it is not regular but only "quasi-regular" **(14**, p. 18**)**. In fact, the graph shows that each of its $336/6 = 56$ vertices is surrounded by three triangles and three squares, arranged alternately.

11. The symmetry group of the polyhedron $(1\ 1\ 1_1^l)^m$. The group 3.3 has an involutory automorphism which interchanges R_1 and R_2. Adjoining a new element T which transforms the group in this manner (so that $T^2 = E$, $R_1 = TR_2T$ and $TR_3 = R_3T$), we obtain the larger group

11.1 $$T^2 = R_2^2 = R_3^2 = (R_2R_3)^3 = (TR_2)^{2l} = (TR_3)^2 = (TR_2R_3)^{2m} = E,$$

which is $G^{3,2l,2m}$ in the notation of **(12**, pp. 104–105**)**. As a transformation of the contravariant coordinates, leaving 3.1 invariant, we may take T to be

$$(x^1, x^2, x^3) \quad \rightarrow \quad (\bar{x}^2, \bar{x}^1, \bar{x}^3).$$

Although this is not a unitary reflection but an "anti-projectivity", its effect on the polyhedron 10.1 is like that of reflecting in the common edge PQ of two adjacent faces. Since the fundamental region of the symmetry group of the polyhedron is one-sixth of a face (shaded in Fig. 1), this symmetry group is precisely $G^{3,2l,2m}$, and $(1\ 1\ 1_1^l)^m$ is the regular skew polyhedron

$$\{3, 2l\}_{2m}$$

(12, p. 127**)** or $\{3, 2l\,|, m\}$ **(11**, p. 59**)**. In particular, we see again that the polyhedron

$$(1\ 1\ 1_1^4)^4 = \{3, 8\}_3 = \{3, 8\,|, 4\}$$

has 42 vertices, 168 edges, and 112 triangular faces.

Setting $l = \frac{1}{2}n$ and $m = \frac{1}{2}p$ in 3.7, we deduce that the even values of n and p for which the group $G^{3,n,p}$ is finite are given by

11.2 $$\cos\frac{4\pi}{n} + \cos\frac{4\pi}{p} < \tfrac{1}{2},$$

in agreement with **(12**, p. 120, Fig. 3**)**, where, however, the locus drawn is not

$$\cos\frac{4\pi}{n} + \cos\frac{4\pi}{p} = \tfrac{1}{2} \quad \text{but} \quad \cos\frac{2\pi}{n} + \cos\frac{2\pi}{p} = \tfrac{1}{2}.$$

One is tempted to conjecture that the new criterion 11.2 will remain valid when n and p are not both even. It is satisfied by all the known finite groups $G^{s,n,p}$, including

$$G^{3,9,9} \simeq LF(2,19), \quad G^{3,8,11} \simeq PGL(2,23), \quad G^{3,7,15} \simeq LF(2,29).$$

The only unknown group that should be finite, according to 11.2, is $G^{3,7,16}$, whose subgroup (2, 3, 7; 8), of index 2, is defined by

$$S^2 = T^2 = (ST)^7 = (S^{-1}TST)^8 = E.$$

Perhaps some enterprising reader will test this with an electronic computer.

12. Barnes's new extreme senary form. Finally, we will show how, when the complex 3-space is regarded as a real 6-space, the vertices of the honeycomb $(1\ 1\ 2_4)^4$ yield the lattice representing Barnes's new quadratic form.

As a basis for the lattice 10.2, where $c = \frac{1}{2}(-1 + i\sqrt{7})$, we may use the six complex vectors

$$\begin{aligned} \mathbf{t}_1 &= (1, \bar{c}, 1), & \mathbf{t}_2 &= (2, 0, 0), \\ \mathbf{t}_3 &= (1, 1, \bar{c}), & \mathbf{t}_4 &= (0, 2, 0), \\ \mathbf{t}_5 &= (\bar{c}, 1, 1), & \mathbf{t}_6 &= (0, 0, 2) \end{aligned}$$

(cf. **18**, p. 397**)**, which combine to yield

$$\begin{aligned} (c,\ \ c, 0) &= -\mathbf{t}_1 - \mathbf{t}_5 + \mathbf{t}_6, \\ (c, -c, 0) &= \mathbf{t}_1 - \mathbf{t}_2 + \mathbf{t}_4 - \mathbf{t}_5, \end{aligned}$$

and so on. Thus the general vector of the lattice is

$$\begin{aligned} \mathbf{z} &= \textstyle\sum u_j \mathbf{t}_j \\ &= (u_1 + 2u_2 + u_3 + \bar{c}u_5,\ \ \bar{c}u_1 + u_3 + 2u_4 + u_5,\ \ u_1 + \bar{c}u_3 + u_5 + 2u_6), \end{aligned}$$

where the u's are real integers; and its norm is

$$\begin{aligned} \mathbf{z}\bar{\mathbf{z}} &= (u_1 + 2u_2 + u_3 + \bar{c}u_5)\ (u_1 + 2u_2 + u_3 + cu_5) \\ &\quad + (\bar{c}u_1 + u_3 + 2u_4 + u_5)\ (cu_1 + u_3 + 2u_4 + u_5) \\ &\quad + (u_1 + \bar{c}u_3 + u_5 + 2u_6)\ (u_1 + cu_3 + u_5 + 2u_6) \\ &= 4(u_1^2 + u_2^2 + u_3^2 + u_4^2 + u_5^2 + u_6^2 \\ &\quad + u_1u_2 + u_2u_3 + u_3u_4 + u_4u_5 + u_5u_6 + u_6u_1) \\ &\quad - 2(u_1u_4 + u_2u_5 + u_3u_6) \qquad 12.1 \\ &= (2u_1 + u_2 - \tfrac{1}{2}u_4 + u_6)^2 + (2u_3 + u_4 - \tfrac{1}{2}u_6 + u_2)^2 \\ &\quad + (2u_5 + u_6 - \tfrac{1}{2}u_2 + u_4)^2 + \tfrac{7}{4}(u_2^2 + u_4^2 + u_6^2). \end{aligned}$$

Since this definite senary form represents 4 and has determinant 7^3, one naturally looks to see whether it is equivalent to the new form $4\phi_6$ discovered by Barnes **(4)**. To establish this equivalence, we note that Barnes represented his form by a real six-dimensional lattice consisting of the points in Euclidean 7-space whose coordinates

$$y_0, y_1, y_2, y_3, y_4, y_5, y_6$$

are integers satisfying

$$\sum_0^6 y_j = 0, \quad \sum_0^6 j y_j \equiv 0 \pmod 7.$$

In terms of unit vectors $\mathbf{e}_j$ along the seven orthogonal axes, he observed that the minimal vectors are

$$\mathbf{e}_a + \mathbf{e}_b - \mathbf{e}_c - \mathbf{e}_d,$$

with distinct suffixes satisfying

$$a + b \equiv c + d \pmod 7$$

(5, §7), and that this set of 42 vectors $\sum y_j \mathbf{e}_j$ is transformed into itself by the permutations

$$y_j \to y_{j+1} \quad \text{and} \quad y_j \to y_{3j},$$

where the suffixes are reduced modulo 7. The latter permutation (of period 6), applied repeatedly to the vector

$$\mathbf{t}_1 = -\mathbf{e}_0 + \mathbf{e}_1 + \mathbf{e}_3 - \mathbf{e}_6,$$

yields

$$\begin{aligned}
\mathbf{t}_2 &= -\mathbf{e}_0 + \mathbf{e}_3 + \mathbf{e}_2 - \mathbf{e}_4, \\
\mathbf{t}_3 &= -\mathbf{e}_0 + \mathbf{e}_2 + \mathbf{e}_6 - \mathbf{e}_5, \\
\mathbf{t}_4 &= -\mathbf{e}_0 + \mathbf{e}_6 + \mathbf{e}_4 - \mathbf{e}_1, \\
\mathbf{t}_5 &= -\mathbf{e}_0 + \mathbf{e}_4 + \mathbf{e}_5 - \mathbf{e}_3, \\
\mathbf{t}_6 &= -\mathbf{e}_0 + \mathbf{e}_5 + \mathbf{e}_1 - \mathbf{e}_2.
\end{aligned}$$

The six $\mathbf{t}$'s, being independent, constitute a basis for the lattice, and enable us to construct the form

$$\begin{aligned}
\Big(\sum_1^6 u_j \mathbf{t}_j\Big)^2 &= \{-\sum_1^6 u_j \mathbf{e}_0 + (u_1 + u_2 - u_4)\mathbf{e}_1 + (u_3 + u_4 - u_6)\mathbf{e}_2 \\
&\qquad + (u_2 + u_3 - u_5)\mathbf{e}_3 + (u_5 + u_6 - u_2)\mathbf{e}_4 \\
&\qquad + (u_6 + u_1 - u_3)\mathbf{e}_5 + (u_4 + u_5 - u_1)\mathbf{e}_6\}^2 \\
&= \Big(\sum_1^6 u_j\Big)^2 + \sum_1^6 (u_j + u_{j+1} - u_{j+3})^2 \\
&= 4\sum_1^6 u_j^2 + 4\sum_1^6 u_j u_{j+1} - 2\sum_1^3 u_k u_{k+3},
\end{aligned}$$

which is 12.1.

We observe that this form (equivalent to Barnes's has the elegant matrix

$$\begin{pmatrix} 4 & 2 & 0 & -1 & 0 & 2 \\ 2 & 4 & 2 & 0 & -1 & 0 \\ 0 & 2 & 4 & 2 & 0 & -1 \\ -1 & 0 & 2 & 4 & 2 & 0 \\ 0 & -1 & 0 & 2 & 4 & 2 \\ 2 & 0 & -1 & 0 & 2 & 4 \end{pmatrix}$$

Barnes (**4**, p. 240) has shown that its group of automorphs is

$$\mathfrak{S}_2 \times PGL(2, 7).$$

In view of §11, we now recognize this as the symmetry group $G^{2,3,3}$ of the complex polyhedron $(1\ 1\ 1_1^4)^4$.

TABLE I

THE COMPUTATION OF c FOR $[1\ 1\ 1^l]^m$ (see 3.5)

l	m	$2\cos\frac{\pi}{l}$	$2\cos\frac{\pi}{m}$	$2\cos\frac{\pi}{i}$	c
3	m	1	$2\cos\pi/m$	$2\cos 2\pi/m$	$\zeta^{2\pi i/m}$
l	3	$2\cos\pi/l$	1	$-2\cos\pi/l$	$-1-\zeta^{-2\pi i/l}$
4	4	$\sqrt{2}$	$\sqrt{2}$	$-1/\sqrt{2}$	$(-1+i\sqrt{7})/2$
4	5	$\sqrt{2}$	τ	$-\tau^{-2}/\sqrt{2}$	$-1-\tau\bar{\omega}$
5	4	τ	$\sqrt{2}$	-1	$\tau\omega$
4	6	$\sqrt{2}$	$\sqrt{3}$	0	$i\sqrt{2}$
6	4	$\sqrt{3}$	$\sqrt{2}$	$-2/\sqrt{3}$	$-1+i\sqrt{2}$

TABLE II

FINITE NON-REAL GROUPS GENERATED BY n INVOLUTORY REFLECTIONS

Group	Number of dimensions n	$\det(2a_{jk})$	Exponents $m_1, m_2, \ldots, m_n$	Number of reflections Σm_i	Order of group $\Pi(m_i+1)$	Order of centre k	$\frac{h}{k}$	Symbol of Shephard (29)	Symbol of Todd (31)
$[1\ 1\ r^l]^3$ $[1\ 1 \ldots 1]^l$	$r+2$	$4\sin^2\frac{\pi}{l}$	$r+1, l-1, 2l-1, \ldots, (r+1)l-1$	$l\binom{n}{2}$	$l^{n-1}n!$	(l, n)	$\frac{l(n-1)}{(l, n)}$	$[1\ 1; r]^l$	$G(l, l, n)$
$[1\ 1\ 1^4]^4$	3	1	3, 5, 13	21	336	2	7	—	No. 24
$[1\ 1\ 1^5]^4$ $[1\ 1\ 1^4]^5$	3	r^{-2}	5, 11, 29	45	2160	6	5	—	No. 27
$[2\ 1\ 1^4]^3$ $[2\ 1\ 1]^4$	4	1	3, 7, 11, 29	40	64·5!	4	5	$[2\ 1; 1]^4$	No. 29
$[2\ 1\ 2]^3$	5	2	3, 5, 9, 11, 17	45	72·6!	2	9	$[2\ 1; 2]^3$	No. 33
$[2\ 1\ 3]^3$	6	1	5, 11, 17, 23, 29, 41	126	108·9!	6	7	$[2\ 1; 3]^3$	No. 34

TABLE III

SOME COMPLEX POLYTOPES

Symbol defined in §10	Number of dimensions	Number of vertices	Symbol of Shephard (29) or of Coxeter (11; 12)	Reference
$(1\ 1\ r_r{}^l)^s$	$n = r + 2$	ln	$(1\ 1; r_r)^l = \beta^l{}_n$	(29, p. 377)
$(1_1\ 1\ r^l)^s$	$n = r + 2$	l^{n-1}	$(1_1\ 1; r)^l = \frac{1}{l}\gamma^l{}_n$	(29, p. 378)
$(1_1\ 1 \ldots 1)^l$	n	l^{n-1}	—	p. 251
$(1\ 1\ 1_1{}^4)^4$	3	42	$\{3, 8\|, 4\} = \{3, 8\}_3$	p. 265
$(1_1\ 1\ 1^4)^4$	3	56	—	p. 266
$(1\ 1\ 1_1{}^5)^4$	3	216	$\{3, 10\|, 4\} = \{3, 10\}_3$	(11, p. 61)
$(1\ 1\ 1_1{}^4)^5$	3	270	$\{3, 8\|, 5\} = \{3, 8\}_{10}$	(11, p. 61)
$(1_1\ 1\ 1^5)^4$	3	360	—	—
$(1_1\ 1\ 1^4)^5$	3	360	—	—
$(2_2\ 1\ 1^4)^3$	4	80	$(2_2\ 1; 1)^4 = (\frac{1}{4}\gamma^4{}_4)^{+1}$	(29, p. 382)
$(2_2\ 1\ 1)^4$	4	80	—	—
$(2\ 1\ 1_1{}^4)^3$	4	160	$(2\ 1; 1_1)^4$	p. 261
$(2\ 1\ 1_1)^4$	4	320	—	—
$(2\ 1\ 2_2)^3$	5	80	$(2\ 1; 2_2)^3 = (\frac{1}{3}\gamma^5{}_4)^{+1}$	(29, p. 381)
$(2\ 1_1\ 2)^3$	5	432	$(2\ 1_1; 2)^3$	(29, p. 381)
$(2\ 1\ 3_3)^3$	6	756	$(2\ 1; 3_3)^3 = (\frac{1}{3}\gamma^6{}_4)^{+2}$	(18, p. 391)
$(2_2\ 1\ 3)^3$	6	4032	$(2_2\ 1; 3)^3 = (\frac{1}{3}\gamma^6{}_4)^{+1}$	(18, p. 391)
$(2\ 1_1\ 3)^3$	6	54432	$(2\ 1_1; 3)^3$	—

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ABSTRACT DEFINITIONS FOR REFLECTION GROUPS

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Introduction. This paper is an appendix to "Finite unitary reflection groups" by J. A. Todd and the author **(3)** in which the irreducible finite groups generated by reflections in unitary space **(2**, p. 364**)** are enumerated and abstract definitions are given for all those n-dimensional groups which may be generated by n reflections. (All real reflection groups are of this kind.) The purpose of this paper is to supply abstract definitions for all the remaining groups, namely those that require more than n reflections to generate them.

Some of the original definitions in our paper have been elegantly modified by Coxeter **(1)**. The definitions given here are constructed in such a way as to form natural extensions of his scheme.

1. The groups $G(m, r, 3)$. Let θ be a primitive mth root of unity, r a divisor of m, and $s = m/r$. Then by permuting the rows (or columns) of the diagonal $n \times n$ matrices

$$\operatorname{diag}\{\theta^{\rho_1}, \theta^{\rho_2}, \ldots, \theta^{\rho_n}\}, \qquad \sum \rho_i \equiv 0 \pmod{r},$$

in every possible way, we obtain a group of $sm^{n-1}n!$ matrices which is denoted by $G(m, r, n)$ **(3**, p. 277**)**. When $r = 1$ or $r = m$, this group is generated by n reflections: $G(m, 1, n)$ is the symmetry group of the polytope γ_n^m **(2**, p. 374**)** and so, following the notation of **(3)**, we denote it by $[\gamma_n^m]$. Also, when $n > 2$, $G(m, m, n) = [\frac{1}{m}\gamma_n^m]$ is $[1\ 1\ (n-2)^m]^3$ in Coxeter's notation **(1**, pp. 248–249**)**.

Consider first the groups $G(m, r, 3)$ with $r \neq 1, m$. The matrices

$$1.1 \qquad \mathbf{P} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ \mathbf{Q} = \begin{pmatrix} 0 & \theta & 0 \\ \theta^{-1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \mathbf{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \ \mathbf{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \theta^r \end{pmatrix}$$

clearly generate $G(m, r, 3)$, and if we denote by P, Q, R, T respectively the corresponding group elements, then the following relations are satisfied:

$$1.2 \qquad P^2 = Q^2 = R^2 = (QR)^3 = (RP)^3 = (PQ)^3 = E,$$

$$1.3 \qquad \begin{cases} (PTP)(RT^{-1}R) = (RT^{-1}R)(PTP) = (QPRP)^r, \\ TQ = QT, \qquad T(PRP) = (PRP)T, \qquad T^s = E. \end{cases}$$

We shall show that these relations form an abstract definition for the group.

Since $(PTP)^s = PT^sP = E$, $(RT^{-1}R)^s = RT^{-s}R = E$, and $PTP \leftrightarrows RT^{-1}R$, we have

$$(QPRP)^m = ((QPRP)^r)^s = (PTP)^s(RT^{-1}R)^s = E,$$

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that is,

1.4 $$(QPRP)^m = E.$$

It is known that relations 1.2 and 1.4 constitute an abstract definition for the group $G(m, m, 3) = [1\,1\,1]^m$ of order $6m^2$ **(1**, p. 248**)**. Denote an arbitrary element of this group by G; each occurrence of G in an expression stands for a (possibly) different element of the group.

By 1.3, T, and therefore T^m, commutes with PRP and so $PT^mPR = RPT^mP$. This implies

$$(QPRP)^m = (PT^mP)(RT^{-m}R) = RPT^mPT^{-m}R,$$

from which $T^mP = GT^m$. Similarly making use of the fact that T commutes with $RPR = PRP$, we get $T^mR = GT^m$.

Let W be any word, i.e. finite product of the elements P, Q, R, T. Then W must be of the form

$$GT^{m_1}GT^{m_2}G \ldots GT^{m_r}G.$$

Let $w(W)$ be the number of factors P, Q, R occurring in all the elements G of W *except* the first. Since

$$T^{m_1}P = GT^{m_1}, \quad T^{m_1}Q = QT^{m_1}, \quad T^{m_1}R = GT^{m_1},$$

whatever letter P, Q, or R follows T^{m_1} in W it is possible to substitute some product of group elements so as to form an equivalent word W' with $w(W) - w(W') \geqslant 1$. Repeating this process and amalgamating powers of T when they become adjacent, W may be reduced, by a finite number of substitutions, to the standard form

$$W^* = GT^u, \qquad 0 \leqslant u \leqslant s - 1,$$

with $w(W^*) = 0$. Hence the given relations define a group of order $(6m^2)s$ which is $G(m, r, 3)$.

For $G(4, 2, 3)$ a simpler definition than that obtained by substituting $m = 4$ and $r = 2$ in 1.3 is given by 1.2 together with the relations

1.5 $$(QPRP)^4 = E,$$

1.6 $$T^2 = (TQ)^2 = (TPRP)^2 = (TR)^4 = P(TRQR)P(TRQR)^{-1} = E.$$

This can be checked by the Todd-Coxeter method **(1a**, Chap. 2**)**.

2. The other groups. Now consider $G(m, r, n)$ $(r|m;\ r \neq 1,\ m;\ n > 3)$ of order $rm^{n-1}n!$. If we introduce $n - 3$ further generators $S_1, S_2, \ldots, S_{n-3}$ which satisfy the relations

2.1 $$\begin{cases} S_i^2 = (PS_i)^2 = (RQRS_i)^2 = (S_iS_j)^2 = (RS_1)^3 = E \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (i, j = 1, 2, \ldots, n-3;\ |i-j| \geqslant 2), \\ (S_iS_{i-1})^3 = (RS_i)^2 = E \qquad\qquad\qquad\qquad (i = 2, 3, \ldots, n-3), \end{cases}$$

then these, together with 1.2, 1.4, form a definition for $G(m, m, n)$ of order

$m^{n-1}n!$. This is the same definition as that of **(2**, pp. 374–5**)** with the notation changed thus:

$$P = P_1, \quad Q = R_1Q_1R_1, \quad R = R_1, \quad S_i = R_{i+1} \qquad (i = 1, 2, \ldots, n-3).$$

The use of these new generators is due to Coxeter **(1**, p. 248**)** who has shown that they simplify the definitions of some of the reflection groups discussed in **(3)**.

Adding now the generator T satisfying 1.3 and $TS_i = S_iT$ $(i = 1, 2, \ldots, n-3)$ we get an abstract definition for $G(m, r, n)$. The proof is very similar to that of §1, remarking that every word can be reduced to the standard form $G'T^u$, where G' is an arbitrary element of $G(m, m, n)$.

This completes the discussion of $G(m, r, n)$ except for the groups where $n = 2$. Here we take as generators P, Q and T corresponding to the matrices

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{Q} = \begin{pmatrix} 0 & \theta \\ \theta^{-1} & 0 \end{pmatrix}, \qquad \mathbf{T} = \begin{pmatrix} 1 & 0 \\ 0 & \theta^r \end{pmatrix}$$

and obtain the relations

$$P^2 = Q^2 = T^s = E, \quad T(PQ) = (PQ)T, \quad PT^{-1}PT = TPT^{-1}P = (PQ)^r,$$

which can be shown to be an abstract definition in the same way as for the the above groups.

The only remaining groups generated by more than n reflections are seven two-dimensional groups for which abstract definitions have already been given **(3**, pp. 281–282; **1**, Table XIII**)** and the group $[(\frac{1}{2}\gamma_3{}^4)^{+1}]$ (no. 31 in the table of **(3**, p. 301**)**).

This group is generated by the matrices (cf. 1.1):

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\mathbf{S} = \tfrac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The corresponding group elements satisfy the relations 1.2, 1.5, 1.6 and

2.2 $\qquad S^2 = (SP)^3 = (SR)^3 = (SQ)^3 = E,$

2.3 $\qquad (ST)^3 = E.$

By §1, the relations 1.2, 1.5, 1.6 form an abstract definition for $G(4, 2, 3)$, while 1.2, 1.5, 2.2 form an abstract definition for $[1\ 1\ 2]^4 = [(\frac{1}{2}\gamma_3{}^4)^{+1}]$ **(2**, p. 374**)**. We assert that 1.2, 1.5, 1.6, 2.2, 2.3 together form an abstract definition for the group $[(\frac{1}{2}\gamma_3{}^4)^{+1}]$. This can readily be verified by the Todd-Coxeter method by enumerating the six cosets of the subgroup $[1\ 1\ 2]^4$.

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DIOPHANTINE APPROXIMATION AND HOROCYCLIC GROUPS

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1. Introduction. Let ω be an irrational number. It is well known that there exists a positive real number h such that the inequality

$$\left|\omega - \frac{a}{c}\right| < \frac{1}{hc^2} \tag{1}$$

has infinitely many solutions in coprime integers a and c. A theorem of Hurwitz asserts that the set of all such numbers h is a closed set with supremum $\sqrt{5}$. Various proofs of these results are known, among them one by Ford **(1)**, in which he makes use of properties of the modular group. This approach suggests the following generalization.

Let Γ be a (real) zonal horocyclic group. That is, Γ is a Fuchsian group of bilinear transformations of the first kind **(4)** having the real axis as principal circle and ∞ as a parabolic fixed point. A bilinear transformation T is defined by

$$w = Tz = \frac{az + b}{cz + d},$$

where a, b, c and d are real numbers such that $ad - bc = 1$, and w and z are complex numbers. It is convenient to use T to denote the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

as well as for the transformation, and to assume that Γ is such that if T belongs to Γ, then so does

$$-T = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}.$$

Since $-T$ gives rise to the same value of w as T, we do not distinguish between T and $-T$.

Since Γ has a parabolic fixed point at infinity and is discrete, there exists a least positive λ such that $U^\lambda \in \Gamma$, where

$$U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U^\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}.$$

The number λ has been called by Petersson the *width* of the cusp at ∞. The transformation U^λ generates a cyclic subgroup of Γ which we call Γ_U. In particular, if $T \in \Gamma$ then $TU^{m\lambda} \in \Gamma$ for any integer m, and $TU^{m\lambda}$ has the same

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first and third entries, namely a and c, as T. Thus a and c are functions of, and are determined by, the left cosets of Γ_U in Γ.

In terms of a zonal horocyclic group Γ we can generalize Diophantine approximation to irrational numbers by asking if there exists a positive $h = h(\Gamma)$ such that (1) is soluble for transformations T in infinitely many left cosets of Γ_U in Γ, where ω is any real number not belonging to a certain countable set, to be specified. We can, for this purpose, rewrite (1) as

$$|\omega - T\infty| < \frac{1}{hc^2}, \tag{2}$$

and it is possible to express the right-hand side in terms of T also, although it is not particularly advantageous to do this. The exceptional countable set will always include the set of all points congruent to ∞ with respect to Γ.

If such a number h exists, we can then ask what is the supremum of all such h, and whether this supremum is attained. If (2) holds for infinitely many left cosets of Γ_U in Γ, the numbers $T\infty$ satisfying (2) will approximate to ω as closely as we please since it is known that $|c|$ is less than any given positive number for only a finite number of cosets of Γ_U **(3**, Satz 2**)**.

In the particular case where Γ is the modular group $\Gamma(1)$, this reduces to the approximation by rationals to an arbitrary irrational number ω, as described at the beginning of the paper. Ford's demonstration of Hurwitz's theorem does not, however, generalize immediately to an arbitrary zonal horocyclic group, since the possible existence of parabolic cycles not congruent to ∞ causes difficulties. Also, as might be expected, horocyclic groups of the second kind, which have an infinity of generators, are more difficult to treat since the set of points congruent to ∞ covers the real axis more thinly. In this paper we consider horocyclic groups of the first kind only (see §2 for definition).

Applications of the results obtained to the modular group and some of its subgroups are given in §7 and compared with results obtained by others. Considering the generality of the methods used in §§2-6, it is somewhat surprising that best possible results are nevertheless obtained in some cases. The horocyclic group approach also sheds some light on the connexion between irrationals which cannot be closely approximated to and hyperbolic transformations.

2. The existence of numbers h. We write $z = x + iy$, where x and y are real and use $\mathfrak{H}$ to denote the finite upper half-plane $y > 0$, and $\mathfrak{A}$ for the real axis $y = 0$, excluding the point at infinity. We suppose that Γ is a zonal horocyclic group and that $\mathfrak{D}$ is a fundamental region for Γ constructed by Ford's isometric circle method **(2**, §35**)** and occupying the strip $\xi < x \leqslant \xi + \lambda$, where λ is the width of the cusp at infinity and ξ is chosen suitably later.

We suppose that Γ is a horocyclic group of the first kind, i.e. that $\mathfrak{D}$ has finite hyperbolic area. This is the case if and only if $\mathfrak{D}$ has a finite number of sides.

Adjacent sides of $\mathfrak{D}$ meet at vertices which are of two kinds. The point at infinity and vertices lying on $\mathfrak{A}$ are called *cusps*. They can be divided into groups of congruent cusps with respect to Γ, called *parabolic cycles*. One such cycle, which may be the only one, consists of the point at infinity alone.

The remaining vertices all lie in $\mathfrak{H}$. They include points at which different isometric circles meet, and possibly elliptic fixed points of period 2 belonging to Γ which lie at a midpoint of a single bounding arc. Such vertices we call *ordinary vertices of the first kind*.

It may happen that some sides of $\mathfrak{D}$ consist of complete semicircles: if this happens it is convenient to regard such a semicircle as two equal sides meeting at a vertex which is the point farthest from $\mathfrak{A}$; such vertices we call *ordinary vertices of the second kind*.

We now choose ξ so that the line $x = \xi$ passes through an ordinary vertex of the first or second kind. Then every vertex of $\mathfrak{D}$ in $\mathfrak{H}$ is of the first or second kind, and $\mathfrak{D}$ is bounded by a finite number of sides which can be grouped in pairs of equal arcs transformable into each other by transformations of Γ. One side of such a pair (if not part of a straight line) is an arc of the isometric circle $\mathfrak{I}(T)$ of a transformation T, and the other is an equal arc of $\mathfrak{I}(T^{-1})$, where $T \in \Gamma$.

Define h_Γ to be twice the minimum distance of the vertices of $\mathfrak{D}$ in $\mathfrak{H}$ from $\mathfrak{A}$. We are now in a position to state

THEOREM 1. *Let Γ be a zonal horocyclic group of the first kind, and let ω be any real number which is not a parabolic fixed point for Γ. Then there are infinitely many left cosets of Γ_U in Γ whose members T satisfy*

$$\left|\omega - \frac{a}{c}\right| = \left|\omega - T\infty\right| < \frac{1}{h_\Gamma c^2} \qquad (c \neq 0). \tag{3}$$

3. Proof of Theorem 1. Let ω be any real number which is not a parabolic fixed point for Γ and let $\mathfrak{L} = \mathfrak{L}(\omega)$ be the straight line through ω which is perpendicular to $\mathfrak{A}$.

For any $T \in \Gamma$ with $T \notin \Gamma_U$ and for any $h > 0$, let $\mathfrak{S}(T, h)$ be the circle

$$\left|cz - a - \frac{i}{hc}\right| = \frac{1}{hc}, \tag{4}$$

which is of radius $1/(hc^2)$ and touches $\mathfrak{A}$ at a/c, but otherwise lies in $\mathfrak{H}$. If $\mathfrak{L}$ cuts $\mathfrak{S}(T, h)$, we have

$$\left|\omega - \frac{a}{c}\right| < \frac{1}{hc^2},$$

and conversely; i.e. (2) holds.

Let $\mathfrak{L}'(h)$ be the line $y = \frac{1}{2}h$; then

$$T\,\mathfrak{L}'(h) = \mathfrak{S}(T, h)$$

if $T \in \Gamma$, $T \notin \Gamma_U$. For $T \in \Gamma_U$ we define $\mathfrak{S}(T, h)$ to be $\mathfrak{L}'(h)$ and it then follows that

$$S\mathfrak{S}(T, h) = \mathfrak{S}(ST, h) = ST\,\mathfrak{L}'(h) = ST\,\mathfrak{S}(E, h)$$

for all S, $T \in \Gamma$, where E is the identical transformation.

Accordingly $\mathfrak{L}$ cuts $\mathfrak{S}(T, h)$ $(T \notin \Gamma_U)$, and so (2) holds, if and only if $T^{-1}\,\mathfrak{L}$ cuts $\mathfrak{L}'(h)$. But $T^{-1}\,\mathfrak{L}$ is a circle centred at a point of $\mathfrak{A}$ and so cuts $\mathfrak{L}'(h)$ if and only if its radius exceeds $\frac{1}{2}h$. Also, if $T^{-1}\,\mathfrak{L}$ cuts $\mathfrak{L}'(h)$ so does $(T')^{-1}\,\mathfrak{L}$ if T' and T are in the same left coset of Γ_U in Γ; for $T^{-1}\,\mathfrak{L}$ and $(T')^{-1}\,\mathfrak{L}$ are equal circles $r\lambda$ apart where r is an integer. Thus the theorem will follow if we show that $\mathfrak{L}'(h_\Gamma)$ is cut by infinitely many circles $T^{-1}\,\mathfrak{L}$ $(T \in \Gamma)$ at points lying in the strip $\xi < x \leqslant \xi + \lambda$.

Since ω is not a parabolic fixed point, the line $\mathfrak{L}$ passes through the interiors of infinitely many fundamental regions $T\mathfrak{D}$ $(T \in \Gamma)$. Let the fundamental regions through whose closures $\mathfrak{L}$ passes be $T_n\mathfrak{D}$ $(n = 1, 2, 3, \ldots)$ taken in order as a point z moves along $\mathfrak{L}$ from ∞ through $\mathfrak{H}$ to $\mathfrak{A}$; if $\mathfrak{L}$ passes through any points congruent to vertices of $\mathfrak{D}$ round which more than two fundamental regions cluster, then some definite ordering of the associated transformations T_n must be determined.

We suppose that Theorem 1 is false. Then there exists an integer N such that $\mathfrak{L}_n = T_n^{-1}\,\mathfrak{L}$ does not cut $\mathfrak{L}'(h_\Gamma)$ for $n > N$. Let k be the maximum number of images of $\mathfrak{D}$ which are grouped round an ordinary vertex of $\mathfrak{D}$. We show first that $\mathfrak{L}_n$ does not pass through an ordinary vertex of $\mathfrak{D}$ if $n > p = N + k$.

For suppose that $\mathfrak{L}_n$ passes through the ordinary vertex A of $\mathfrak{D}$ for some $n > p$. Then A is at a distance $\frac{1}{2}h_\Gamma$ from $\mathfrak{A}$ and is that point in $\mathfrak{H}$ of $\mathfrak{L}_n$ which is farthest from $\mathfrak{A}$, i.e. the *summit* of $\mathfrak{L}_n$. It follows that A is not an ordinary vertex of the second kind since, if it were, then $\mathfrak{L}_n$ would cut $\mathfrak{A}$ in two parabolic fixed points which is impossible since ω is not a parabolic fixed point.

Thus A must be an ordinary vertex of the first kind, and so lies on an isometric circle $\mathfrak{I}(T)$ of a transformation T of Γ, of which a finite arc γ forms one of the sides of $\mathfrak{D}$. If T is chosen in its right coset of Γ_U in Γ so that TA is also a vertex of $\mathfrak{D}$, then $T\mathfrak{L}_n = TT_n^{-1}\,\mathfrak{L}$ meets the closure of $\mathfrak{D}$ in TA, and so $TT_n^{-1} = T_m^{-1}$ for some $m \neq n$. Moreover $m > n - k > N$. Hence $\mathfrak{L}_m = T\,\mathfrak{L}_n$ does not cut $\mathfrak{L}'(h_\Gamma)$ but meets the closure of $\mathfrak{D}$ in TA which must therefore be the summit of $\mathfrak{L}_m$. We thus have two equal circles $\mathfrak{L}_n$ and $T\,\mathfrak{L}_n$ with summits A and TA. Since the transformation T is equivalent to an inversion in $\mathfrak{I}(T)$ followed by a reflexion in the radical axis of $\mathfrak{I}(T)$ and $\mathfrak{I}(T^{-1})$, this is possible only if A lies on $\mathfrak{I}(T)$ and if $\mathfrak{L}_n = \mathfrak{I}(T)$. Hence γ is an arc of $\mathfrak{L}_n$ with one endpoint at its summit A. The other endpoint is either an ordinary vertex of the first kind which is nearer to $\mathfrak{A}$ than A, or is a parabolic fixed point which is congruent to ω since it is not congruent to ∞. In either case we have a contradiction.

Hence $\mathfrak{L}_n$ does not pass through an ordinary vertex of $\mathfrak{D}$ if $n > p$. Thus, for each $n > p$, $\mathfrak{L}_n$ passes through the interior of $\mathfrak{D}$ cutting its sides at interior

points. This is a contradiction if there are no parabolic cycles other than that containing the point at infinity, and so the theorem is true in this case. Suppose therefore that other parabolic cycles exist. Then, for each $n \geqslant p$, $\mathfrak{L}_n$ contains in its interior a cusp A_n of $\mathfrak{D}$, and consists in part of an arc lying in the interior of $\mathfrak{D}$ whose endpoints B_n, B_n' lie on sides l_n, l_n' of $\mathfrak{D}$ which meet at A_n. Here

$$T_n B_n' = T_{n+1} B_{n+1},$$

and the points B_n, B_n' are interior points of the sides l_n, l_n', respectively. It follows that

$$T_{n+1}^{-1} T_n$$

transforms l_n' into l_{n+1} and therefore maps A_n into A_{n+1}. Hence all the points A_n for $n \geqslant p$ are congruent and so belong to the same parabolic cycle. Accordingly there exists a positive integer q such that

$$A_{p+rq} = A_p \qquad (r = 0, 1, 2, \ldots)$$

and

$$T_{p+rq}^{-1} T_p = P^r, \qquad (r = 0, 1, 2, \ldots)$$

where

$$P = T_{p+q}^{-1} T_p .$$

Here P is a parabolic transformation having A_p as fixed point.

Choose a bilinear transformation S_1 having $\mathfrak{A}$ as fixed circle and such that $S_1 A_p = \infty$. Then $P = S_1^{-1} U^{\mu} S_1$ for some $\mu \neq 0$, and so

$$S_1 \mathfrak{L}_{p+rq} = S_1 P^r T_p^{-1} \mathfrak{L} = U^{\mu r} S_1 T_p^{-1} \mathfrak{L} = U^{\mu r} S_1 \mathfrak{L}_p. \tag{5}$$

Since $\mathfrak{L}_{p+rq}$ cuts l_p and l_p', $S_1 \mathfrak{L}_{p+rq}$ cuts $S_1 l_p$ and $S_1 l_p'$; these are lines perpendicular to $\mathfrak{A}$ since they pass through $S_1 A_p = \infty$, and this holds for $r = 0, 1, 2, \ldots$. But, by (5), the set of circles $S_1 \mathfrak{L}_{p+rq}$ consists of the circle $S_1 \mathfrak{L}_p$ and its translates through multiples of μ parallel to $\mathfrak{A}$, and so $S_1 \mathfrak{L}_{p+rq}$ cannot cut $S_1 l_p$ and $S_1 l_p'$ for every $r \geqslant 0$.

This contradiction shows that no integer N exists, and this completes the proof of Theorem 1.

4. Upper and lower bounds for sup h. Let $\mathfrak{E}(E, \Gamma)$ denote the set of all positive numbers h for which (2) holds for transformations T belonging to infinitely many left cosets of Γ_U in Γ, where Γ is, as previously, a zonal horocyclic group of the first kind. The reason for displaying E will appear later. By Theorem 1, $h_\Gamma \in \mathfrak{E}(E, \Gamma)$ and so $h \in \mathfrak{E}(E, \Gamma)$ for all positive $h \leqslant h_\Gamma$. In this section we obtain an upper bound for $\mathfrak{E}(E, \Gamma)$. We prove first

THEOREM 2. *The set* $\mathfrak{E}(E, \Gamma)$ *is bounded above.*

Proof. The group Γ contains hyperbolic transformations since if $T \in \Gamma$

and $c \neq 0$, then $U^rT \in \Gamma$ for all integers r and is hyperbolic if r is sufficiently large. Let

$$H = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{6}$$

be a hyperbolic transformation belonging to Γ with $C > 0$ and $A + D > 2$: this latter condition can always be satisfied by taking H^{-1} if it is not true for H. Write

$$A + D = 2 \sec \theta \qquad (0 < \theta < \tfrac{1}{2}\pi).$$

Then the fixed points of H are

$$z_1 = \{\tan(\tfrac{1}{4}\pi - \tfrac{1}{2}\theta) - D\}/C, \quad z_2 = \{\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\theta) - D\}/C.$$

They are the points F_1, F_2 of Fig. 1.

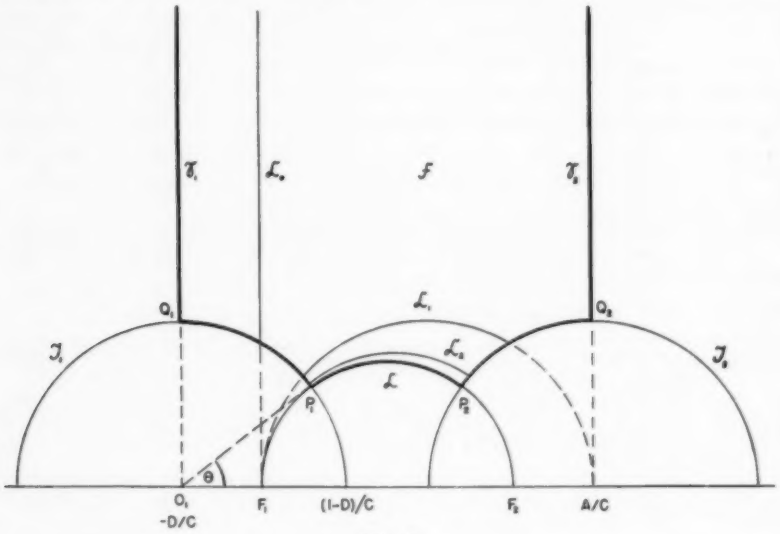

Fig. 1

A fundamental region $\mathfrak{D}_H$ in $\mathfrak{H}$ for the subgroup Γ_H of Γ generated by H is the region exterior to the circles

$$\mathfrak{I}_1 = \mathfrak{I}(H): \quad |Cz + D| = 1,$$

and

$$\mathfrak{I}_2 = \mathfrak{I}(H^{-1}): \quad |Cz - A| = 1.$$

The circles $H^n\mathfrak{I}_1$ ($n = 0, \pm 1, \pm 2, \ldots$) bound the images of $\mathfrak{D}_H$ by Γ_H, and consist of a set of non-intersecting coaxial circles with limit points F_1 and F_2. For $n > 1$, these circles are interior to $H\mathfrak{I}_1 = \mathfrak{I}_2$, and all contain F_2; for $n < 0$, these circles are interior to $\mathfrak{I}_1$ and all contain F_1.

Let $\mathfrak{C}$ be the semicircle in $\mathfrak{H}$ on F_1F_2 as diameter. $\mathfrak{C}$ has radius $(\tan\theta)/C$ and cuts $\mathfrak{J}_1$ and $\mathfrak{J}_2$ orthogonally at the points P_1 and P_2, say; θ is the angle $P_1O_1F_1$, where O_1 is the centre of $\mathfrak{J}_1$.

Suppose that m is the least positive value of $|c|$ for all transformations $T \in \Gamma$; by considering $T^{-1}U^{\lambda}T$ we see that $m \geqslant 1/\lambda$. Take any number h satisfying

$$\begin{aligned} h &> \max\left(\frac{2\tan\theta}{C}, \frac{C}{m^2}\cot\tfrac{1}{2}\theta\right) \\ &= \frac{[(A+D)^2-4]^{\frac{1}{2}}}{|C|}\max\left\{1, \frac{C^2}{m^2(|A+D|-2)}\right\} \\ &= \rho_H, \end{aligned} \tag{7}$$

say. Also take $\omega = z_1$. We show that $\mathfrak{L}(\omega)$ cuts only a finite number of the circles or straight lines $\mathfrak{S}(T, h)$ for $T \in \Gamma$.

If $\mathfrak{L}$ cuts $\mathfrak{S}(T, h)$ at a point of $\mathfrak{H}$ lying in $H^{-n}\mathfrak{D}_H$, then $H^n\mathfrak{L}$ will cut

$$H^n\mathfrak{S}(T, h) = \mathfrak{S}(H^nT, h)$$

in a point of $\mathfrak{H}$ lying in $\mathfrak{D}_H$. But for $n < 0$, $H^n\mathfrak{L}$ lies entirely within $\mathfrak{J}_1$, and so does not meet $\mathfrak{D}_H$; thus we need only consider $n \geqslant 0$. Let $\mathfrak{C}_n$ be the closure of the part of $H^n\mathfrak{L}$ which lies in $\mathfrak{D}_H$. Then it suffices to show that the arcs $\mathfrak{C}_n$ $(n = 0, 1, 2, \ldots)$ have only a finite number of intersections with the curves $\mathfrak{S}(T, h)$ for $T \in \Gamma$.

For $n \geqslant 1$, $H^n\mathfrak{L}$ is a circle orthogonal to $\mathfrak{A}$ passing through the fixed point ω and through $\zeta_n = H^n\infty$. The point ζ_n lies within $H^n\mathfrak{J}_1$, and so within $\mathfrak{J}_2$, and $z_2 < \zeta_n \leqslant \zeta_1 = A/C$. Hence the arcs $\mathfrak{C}_n$ for $n \geqslant 0$ all lie in the closed region $\mathfrak{F}$ of $\mathfrak{H}$ lying between the abscissae $x = -D/C$, $x = A/C$ and outside $\mathfrak{J}_1$, $\mathfrak{J}_2$ and $\mathfrak{C}$. Let Q_1, Q_2 be the summits of $\mathfrak{J}_1$ and $\mathfrak{J}_2$ respectively so that the boundary of $\mathfrak{F}$ consists of five sides (taken closed) joining the vertices ∞, Q_1, P_1, P_2, Q_2 and ∞. We write $\mathfrak{S}_1$ and $\mathfrak{S}_2$ for the sides joining ∞ to Q_1 and Q_2 respectively.

We divide the proof into five cases.

(i) The line $\mathfrak{L}'(h) = \mathfrak{S}(E, h)$ cuts only a finite number of the arcs $\mathfrak{C}_n$, since $\mathfrak{C}_n$ tends to $\mathfrak{C}$ as $n \to \infty$ and $\frac{1}{2}h$ exceeds $(\tan\theta)/C$ which is the radius of $\mathfrak{C}$.

We now show that no circle $\mathfrak{S}(T, h)$, for $T \in \Gamma$, $T \notin \Gamma_U$, has any points in common with $\mathfrak{F}$.

Suppose (ii) that $\mathfrak{S}(T, h)$ meets $\mathfrak{S}_1$ but contains no point of the arc Q_1P_1. Then its radius exceeds $1/C$, so that

$$C > hc^2 \geqslant hm^2 > C\cot\tfrac{1}{2}\theta,$$

which is a contradiction since $\theta < \frac{1}{2}\pi$. Similarly $\mathfrak{S}(T, h)$ cannot meet $\mathfrak{S}_2$ and contain no point of Q_2P_2.

Write $T_1 = HT$, where

$$T_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \tag{8}$$

and suppose next (iii) that $c_1 = Ca + Dc = 0$, so that $\mathfrak{S}(T, h)$ touches $\mathfrak{A}$ at O_1. Then $T_1 \in \Gamma_U$, so that we may take $c = C$, $a = -D$. Then if $\mathfrak{S}(T, h)$ meets $\mathfrak{F}$, $2/(hc^2) \geqslant 1/C$ so that, by (7),

$$\tan\theta < 1, \quad C^2 < 2m^2 \tan\tfrac{1}{2}\theta.$$

This gives a contradiction since $2\tan\frac{1}{2}\theta < 2(\sqrt{2}-1) < 1$ and $C^2 \geqslant m^2$. In the same way we can show that $\mathfrak{S}(T, h)$ cannot touch $\mathfrak{A}$ at A/C.

We now suppose (iv) that $c_1 \neq 0$ and that $\mathfrak{S}(T, h)$ contains in its interior or on its perimeter a point

$$z = (e^{i\phi} - D)/C$$

of Q_1P_1, where $\theta \leqslant \phi \leqslant \frac{1}{2}\pi$. Then, by (4),

$$\left|cz - a - \frac{i}{hc}\right| = \frac{1}{C}\left|(c\cos\phi - c_1) + i\left(c\sin\phi - \frac{C}{hc}\right)\right| < \frac{1}{hc}.$$

Hence

$$\begin{aligned}\frac{2C}{h}\sin\phi &> (c\cos\phi - c_1)^2 + c^2\sin^2\phi \\ &= (c + c_1)^2\sin^2\tfrac{1}{2}\phi + (c - c_1)^2\cos^2\tfrac{1}{2}\phi.\end{aligned}$$

Since $T_1 \in \Gamma$ and $c_1 \neq 0$, either $|c + c_1| \geqslant 2m$ or $|c - c_1| \geqslant 2m$. Also $\sin^2\frac{1}{2}\phi \leqslant \cos^2\frac{1}{2}\phi$, so that

$$\frac{2C}{h}\sin\phi \geqslant 4m^2\sin^2\tfrac{1}{2}\phi.$$

Thus

$$h \leqslant \frac{C}{m^2}\cot\tfrac{1}{2}\phi < \frac{C}{m^2}\cot\tfrac{1}{2}\theta,$$

which contradicts (7). We can also show that $\mathfrak{S}(T, h)$ cannot contain a point of Q_2P_2; for, if it did, $\mathfrak{S}(H^{-1}T, h)$ would contain a point of Q_1P_1.

Finally (v) suppose that $\mathfrak{S}(T, h)$ contains a point of P_1P_2 but not any point of Q_1P_1 or Q_2P_2. Then the radius of $\mathfrak{S}(T, h)$ is greater than that of the circle which touches $\mathfrak{A}$ and touches $\mathfrak{C}$ internally at P_1. This circle has radius $(\tan\frac{1}{2}\theta)/C$, so that

$$1/(hc^2) > \tan\tfrac{1}{2}\theta/C.$$

Thus

$$h < (C/c^2)\cot\tfrac{1}{2}\theta < (C/m^2)\cot\tfrac{1}{2}\theta,$$

which contradicts (7).

We have therefore shown that no circle $\mathfrak{S}(T, h)$ with $T \in \Gamma$, $T \notin \Gamma_U$, has any point in common with $\mathfrak{F}$, so that $\mathfrak{L}$ cuts at most a finite number of the circles $\mathfrak{S}(T, h)$ and therefore $h \notin \mathfrak{E}(E, \Gamma)$. This proves Theorem 2.

If we define

$$h(E, \Gamma) = \sup h \qquad (h \in \mathfrak{E}(E, \Gamma)),$$

and write

$$h_\Gamma' = \inf \rho_H \qquad (H \text{ hyperbolic in } \Gamma), \tag{9}$$

we have proved

THEOREM 3. $h_\Gamma \leqslant h(E, \Gamma) \leqslant h_\Gamma'$.

5. Approximation to ω by $ST\infty$ for fixed S. Let $S \in \Omega_R$, where Ω_R is the continuous group of all bilinear transformations with $\mathfrak{A}$ as principal circle, and

$$S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \tag{10}$$

We consider whether it is possible to approximate to a real number ω by $ST\infty$ where T belongs to a zonal horocyclic group Γ of the first kind. This is not quite the same as approximating to $S^{-1}\omega$ by $T\infty$, since instead of using $1/c^2$ as a measure of approximation we shall use $1/c_1^2$ where $T_1 = ST$ and is given by (8).

Let $\mathscr{R}(\Gamma_U, \Gamma)$ be a set of representatives of the left cosets of Γ_U in Γ arranged in such an order that $|c| \to \infty$ as T runs through $\mathscr{R}(\Gamma_U, \Gamma)$. We prove the

LEMMA. *Let $\mathscr{R}'$ be any subset of $\mathscr{R}(\Gamma_U, \Gamma)$ consisting of transformations T for which $T_1\infty = ST\infty$ is bounded. Then $|c_1| < M$ for only a finite number of $T \in \mathscr{R}'$ where M is any given positive number.*

Proof. Suppose that $|T_1\infty| \leqslant K$ for all $T \in \mathscr{R}'$. Then

$$\begin{aligned} |c_1| &= |\gamma a + \delta c| = |c||\gamma T\infty + \delta| = |c|/|\gamma T_1\infty - \alpha| \\ &> |c|/\{|\gamma|K + |\alpha|\}. \end{aligned}$$

Since $|c| < M\{|\gamma|K + |\alpha|\}$ for only a finite number of $T \in \mathscr{R}(\Gamma_U, \Gamma)$, the result follows.

Now suppose that ω is any finite real number such that $S^{-1}\omega$ is not a parabolic point of Γ. Then, for any $h \in \mathfrak{E}(E, \Gamma)$,

$$\left| S^{-1}\omega - T\infty \right| < \frac{1}{hc^2} \qquad (c \neq 0), \tag{11}$$

for infinitely many $T \in \mathscr{R}(\Gamma_U, \Gamma)$. Let $\mathscr{R}'$ be an infinite subset of $\mathscr{R}(\Gamma_U, \Gamma)$ for which (11) holds. Since $|c| \to \infty$ as T runs through $\mathscr{R}(\Gamma_U, \Gamma)$, we may suppose that $\mathscr{R}'$ is taken so that the points $T\infty$, for $T \in \mathscr{R}'$, lie in as small an interval as we please round $S^{-1}\omega$, and hence that the points $ST\infty$ lie in a bounded region containing ω. It follows from the lemma that $|c_1| < M$ for a finite number of $T \in \mathscr{R}'$ only, and we may therefore arrange the transformations T in $\mathscr{R}'$ in such an order that both $|c|$ and $|c_1|$ tend to infinity as T runs through $\mathscr{R}'$.

For any $T \in \mathscr{R}'$ we have, from (11),

$$|\omega - ST\infty| = |(S^{-1}\omega - T\infty)\{(\gamma S^{-1}\omega + \delta)(\gamma T\infty + \delta)\}^{-1}|$$
$$< \frac{1}{h}\left|\frac{\gamma T\infty + \delta}{\gamma S^{-1}\omega + \delta}\right| \; |c(\gamma T\infty + \delta)|^{-2}$$
$$= \frac{1}{hc_1^2}\left|\frac{\gamma T\infty + \delta}{\gamma S^{-1}\omega + \delta}\right|.$$

As T runs through $\mathscr{R}'$,

$$\left|\frac{\gamma T\infty + \delta}{\gamma S^{-1}\omega + \delta}\right| \to 1,$$

and so, if h' is any positive number less than h, we have

$$|\omega - ST\infty| < \frac{1}{h'c_1^2}$$

for infinitely many $T \in \mathscr{R}(\Gamma_U, \Gamma)$. We have therefore proved

THEOREM 4. *If $h < h(E, \Gamma)$, $S \in \Omega_R$, and ω is any finite real number such that $S^{-1}\omega$ is not a parabolic fixed point of Γ, then*

$$|\omega - ST\infty| = |\omega - T_1\infty| < \frac{1}{hc_1^2}$$

for all T belonging to an infinite subset $\mathscr{R}'$ of $\mathscr{R}(\Gamma_U, \Gamma)$ which may be arranged so that $|c_1| \to \infty$ as T runs through $\mathscr{R}'$.

Let $\mathfrak{E}(S, \Gamma)$ be the set of all positive h for which the conclusions of Theorem 4 hold with the conditions there stated on S, ω and $\mathscr{R}'$, and let $h(S, \Gamma)$ be its supremum. Then, by Theorem 4,

$$h(S, \Gamma) \geqslant h(E, \Gamma).$$

But we have, conversely,

$$\left|S^{-1}\omega - T\infty\right| < \frac{1}{hc^2}\left|\frac{\gamma S^{-1}\omega + \delta}{\gamma T\infty + \delta}\right|$$

for every h in $\mathfrak{E}(S, \Gamma)$. Since $ST\infty \to \omega$ as T runs through $\mathscr{R}'$, $T\infty \to S^{-1}\omega$ as T runs through $\mathscr{R}'$ and so

$$h(E, \Gamma) \geqslant h(S, \Gamma).$$

We have therefore proved

THEOREM 5. *If $S \in \Omega_R$, then $h(E, \Gamma) = h(S, \Gamma)$.*

Hence all the sets $\mathfrak{E}(S, \Gamma)$ are identical, with the possible exception of one point.

We can also prove

THEOREM 6. *$h_\Gamma \in \mathfrak{E}(S, \Gamma)$ if $S \in \Omega_R$.*

To prove this we carry through the argument of §3 with certain modifications. We wish to show that $S^{-1}\,\mathfrak{L}$ cuts $\mathfrak{S}(T, h)$ for T in infinitely many left cosets of Γ_U in Γ, when $S^{-1}\omega$ is not a parabolic fixed point. The demonstration proceeds as in §4 with $\mathfrak{L}$ replaced by $S^{-1}\,\mathfrak{L}$ wherever it occurs after (4). Where before we used the fact that ω was not a parabolic fixed point we now require that $S^{-1}\omega$ is not a parabolic fixed point. As before, $S^{-1}\,\mathfrak{L}$ will pass through the closures of infinitely many fundamental regions $T_n\,\mathfrak{D}$ $(n = 1, 2, 3, \ldots)$. We now choose $T_1(\mathfrak{D})$ to be the fundamental region containing the summit of $S^{-1}\,\mathfrak{L}$ and proceed along $S^{-1}\,\mathfrak{L}$ towards $S^{-1}\omega$. The case where the other endpoint of γ lies on $\mathfrak{A}$ can now be excluded since, because of this choice this endpoint would have to be $S^{-1}\omega$ which is impossible as $S^{-1}\omega$ is not a cusp.

6. Another upper bound for h. Suppose that Γ^* is a zonal horocyclic group of the first kind and that Γ is a subgroup of finite index in Γ^*, and so also a zonal horocyclic group of the first kind. Let μ be the width of the cusp at infinity for Γ^*, so that $\lambda = k\mu$, where k is a positive integer, and let $\Gamma_U{}^*$ be the subgroup of Γ^* generated by U^n.

The parabolic fixed points of Γ^* are parabolic fixed points of Γ, and conversely. We suppose that ω is not a parabolic fixed point for Γ. Then, by Theorem 5,

$$h(E, \Gamma) = h(S, \Gamma) \text{ for all } S \in \Gamma^*. \tag{12}$$

Take a fixed $S \in \Gamma^*$ and a fixed hyperbolic H in Γ^* with entries given by (6), such that

$$S \notin \Gamma_H^*\,\Gamma_U^*\,\Gamma, \tag{13}$$

where $\Gamma_H{}^*$ is the subgroup of Γ^* generated by H. Further, let m_{HS} be the least value of $|c|$ for all $T \in \Gamma_H{}^* S\Gamma$, so that $m_{HS} > 0$, by (13). Also take any h with

$$h > \frac{C}{m_{HS}^2} \cot \tfrac{1}{2}\theta = \frac{C}{m_{HS}^2}\left[\frac{|A + D| + 2}{|A + D| - 2}\right]^{\frac{1}{2}} = \rho'_{HS}, \tag{14}$$

where the notation is that used in §4.

Take $\omega = z_1$. We apply the method of §4 to show that $\mathfrak{L}$ does not intersect any $\mathfrak{S}(ST', h)$ for $T' \in \Gamma$. This is the same as showing that no circle[1] $\mathfrak{S}(T, h)$ intersects any arc $\mathfrak{C}_n$ $(n > 0)$ where $T = H^nST'$.

Case (i) of §4 does not arise, and the proof of case (ii) is similar, with m replaced by m_{HS}. Case (iii) also does not arise, since if $c_1 = 0$, where $T_1 = HT$, we should have $T_1 = H^{n+1}ST' \in \Gamma_U{}^*$, which contradicts (13). Similarly if $Ca - Ac = 0$. The proof of cases (iv) and (v) is similar with m replaced by m_{HS}. Thus we have shown that if h satisfies (14) and S satisfies (13) then $h \notin \mathfrak{E}(S, \Gamma)$.

Define

$$h''_\Gamma = \inf \rho'_{HS}$$

[1] $\mathfrak{S}(T, h)$ cannot be a straight line by (13).

for all hyperbolic transformations $H \in \Gamma^*$ and $S \in \Gamma^*$ for which (13) holds; if no such H and S exist define $h_\Gamma'' = \infty$. It follows from (12) and what we have just proved that the following result holds.

THEOREM 7. *$h(S^*, \Gamma) \leqslant h_\Gamma''$ for all $S^* \in \Gamma^*$.*

7. Applications.

7.1. If we take Γ to be the full modular group $\Gamma(1)$ and ω irrational, we obtain $h_\Gamma = \sqrt{3}$. By taking

$$H = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

in §4 we see that if $h > \sqrt{5}$, then $h \notin \mathfrak{E}(E, \Gamma(1))$. Thus

$$\sqrt{3} \leqslant h(E, \Gamma(1)) \leqslant \sqrt{5}.$$

In fact $h(E, \Gamma) = \sqrt{5}$ as can be obtained by a more detailed study of the modular configuration, as shown by Ford **(1)**.

7.2. If we take $\Gamma = \Gamma(2)$, the principal congruence group of level 2 consisting of matrices $T \equiv E \pmod 2$, we obtain by Ford's method the fundamental region $\mathfrak{D}$ consisting of points in the strip $-\frac{1}{2} \leqslant x \leqslant 3/2$, $y > 0$ which lie outside the circles $|2z \pm 1| = 1$, $|2z - 3| = 1$. Thus $h_\Gamma = 1$.

Write

$$V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{15}$$

and take $\Gamma^* = \Gamma(1)$ in §6. Let

$$S = UVU = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad H = H_n = \begin{pmatrix} 2n & 4n^2 - 1 \\ 1 & 2n \end{pmatrix},$$

for any positive integer n. Then (13) states that $S \not\equiv E, V, U, VU \pmod 2$ which is correct, and $m_{HS} = 1$. Hence

$$\rho'_{HS} = \left(\frac{2n+1}{2n-1}\right)^{\frac{1}{2}} \to 1$$

as $n \to \infty$. Thus $h_\Gamma'' \leqslant 1$, and so, by Theorems 3, 5, 6 and 7,

$$h(S^*, \Gamma(2)) = 1, \quad 1 \in \mathfrak{E}(S^*, \Gamma(2)),$$

for all $S^* \in \Gamma$ (1). By taking (i) $S^* = E$, (ii) $S^* = UVU$, (iii) $S^* = V$, we deduce that (1) is soluble for every irrational ω for an infinity of coprime integers a, c in each of the three cases (i) a odd, c even, (ii) a, c both odd, (iii) a even, c odd, if and only if $h \leqslant 1$. These results are due to Scott **(5)**, who also obtained them by considering $\Gamma(2)$.

7.3. Take Γ to be $\Gamma_V(2)$, the group of matrices T congruent to E or V modulo 2. A fundamental region for $\Gamma_V(2)$ is given by

$$0 < x < 2, \quad y > 0, \quad |z| > 1, \quad |z-2| > 1,$$

so that $h_\Gamma = 2$.

As before, take $\Gamma^* = \Gamma(1)$ and

$$S = UVU, \quad H = H_n' = \begin{pmatrix} 2n+1 & 2n(n+1) \\ 2 & 2n+1 \end{pmatrix}$$

for any positive integer n. Then (13) states that $S \not\equiv E, V, U, UV \pmod 2$, which is correct, and $m_{HS} = 1$. Thus $\rho_{HS}' \to 2$ as $n \to \infty$, so that $h_\Gamma'' < 2$. Hence, by Theorems 3, 5, 6 and 7,

$$h(S^*, \Gamma_V(2)) = 2, \quad 2 \in \mathfrak{E}(S^*, \Gamma_V(2))$$

for all $S^* \in \Gamma(1)$.

By taking S^* to be (a) U, (b) E, (c) UVU, we deduce that (1) is soluble for irrational ω for an infinity of coprime integers a, c if and only if $h < 2$, in each of the following three cases (a) (i) or (ii), (b) (i) or (iii), (c) (ii) or (iii) (see §7.2). These results are also due to Scott **(5)**.

7.4. Let Γ be the principal (inhomogeneous) congruence group $\bar{\Gamma}(N)$ of level $N > 2$. This is a self-conjugate subgroup of $\bar{\Gamma}(1)$ of index $\mu(N)$, say. From a set of $\mu(N)$ representatives of the cosets of $\bar{\Gamma}(N)$ in $\bar{\Gamma}(1)$ we can choose a set G_N of $\mu(N)/N$ matrices S^* with different α, γ modulo N and $(\alpha, \gamma, N) = 1$. Let S be such a matrix with

$$\alpha \equiv 0 \pmod N, \quad \gamma = [\tfrac{1}{2}(N-1)]$$

(integral part) so that $(\gamma, N) = 1$. Also take $\Gamma^* = \bar{\Gamma}(1)$ and

$$H = H_n'' = \begin{pmatrix} Nn+1 & Nn(Nn+2) \\ 1 & Nn+1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \pmod N.$$

Then

$$H^r \equiv \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}, \quad H^r U^s \equiv \begin{pmatrix} 1 & s \\ r & rs+1 \end{pmatrix} \pmod N$$

and $S \not\equiv H^r U^s \pmod N$, showing that (13) is satisfied. Also, if $T \in \Gamma_H{}^* S \Gamma$, $c \equiv \gamma \pmod N$, so that $m_{HS} = \gamma$. We have, by (14), $\rho_{HS}' \to 1/\gamma^2$ as $n \to \infty$, and so, by Theorem 7,

$$h(S^*, \bar{\Gamma}(N)) < \{[\tfrac{1}{2}(N-1)]\}^{-2} \sim 4N^{-2}$$

for large N and $S^* \in G_N$.

Lower bounds for $h(S^*, \bar{\Gamma}(N))$ may be obtained from Theorem 3, by estimating h_Γ, for example by **(4**, Theorem 10**)**, but this does not yield good results, at any rate for large N.

Each $S^* \in G_N$ yields a result on the approximation to irrational ω by fractions a/c with $a \equiv \alpha^*$, $c \equiv \gamma^* \pmod N$, where a and c need not be positive integers. A considerable amount of work has been done by Descombes and Poitou, Hartman, Koksma and Tornheim **(6)** on this and related problems

using different methods such as continued fractions. See **(6)** for references. In particular, it is known that $h(S^*, \Gamma(N)) > 4/N^2$ and that $h(S^*, \Gamma(N)) \sim 4/N^2$ for large N. For $N = 3, 4$, $h(S^*, \Gamma(N)) = \sqrt{(5/3)}$ and $\frac{1}{2}$, respectively.

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EQUIVALENT ABELIAN GROUPS

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1. Introduction. Throughout this note all groups are *abelian*, written additively. We refer to Kurosh **(8; 9)** for notation, terminology and theorems used without reference. We recall the notion of a *serving subgroup* (or pure subgroup) $\mathfrak{S}$ of a group $\mathfrak{G}$. This is a subgroup $\mathfrak{S}$ in which for every natural number n every equation $nx = s$, $s \in \mathfrak{S}$ can be solved provided that it can be solved in $\mathfrak{G}$. If $\mathfrak{G}$ is torsion-free, "linearly closed" subgroups coincide with serving subgroups and $\mathfrak{S}$ is a serving subgroup if and only if $\mathfrak{G}/\mathfrak{S}$ is torsion-free. Direct summands are serving subgroups but, in general, the converse is untrue (cf. **4**).

We call the groups $\mathfrak{G}$ and $\mathfrak{H}$ *equivalent* (or *equivalent by subgroups*) if each is isomorphic to a subgroup of the other, i.e.

$$\begin{aligned} \mathfrak{G} \to \phi\mathfrak{G} = \mathfrak{G}' \subset \mathfrak{H}, \\ \mathfrak{H} \to \psi\mathfrak{H} = \mathfrak{H}' \subset \mathfrak{G}, \end{aligned} \tag{1}$$

where ϕ and ψ are isomorphic maps. If $\mathfrak{G}'$ and $\mathfrak{H}'$ are serving subgroups or direct summands of $\mathfrak{H}$ and $\mathfrak{G}$ respectively, we call $\mathfrak{G}$ and $\mathfrak{H}$ *equivalent by serving subgroups or equivalent by direct summands.*

We give a short survey of the main results obtained so far (a "positive answer" means that equivalence implies isomorphism of the groups under consideration, otherwise we speak of a "negative answer").

Equivalence by subgroups. In the main the problem is solved. In general, the answer is negative. De Bruijn (orally) and Kaplansky **(6)** gave a counter-example for periodic groups. Here is one for the case of torsion-free groups (by a slight alteration the groups can be made countable).

Example (i). $\mathfrak{G}$ is the additive group of real numbers, $\mathfrak{H} = \mathfrak{G} + \mathfrak{C}$, where $\mathfrak{C}$ is an infinite cyclic group. $\mathfrak{H}$ can be embedded isomorphically in $\mathfrak{G}$, just as any torsion-free group with at most continuously many elements. So $\mathfrak{G}$ and $\mathfrak{H}$ are equivalent but clearly not isomorphic.

Nevertheless, in each of the following cases, the answer is positive: the class of complete groups,[1] the class of groups with a finite number of generators (cf. **1**), the class of subgroups of the additive group of rationals.

Equivalence by serving subgroups. In the main the problem is solved. In general, the answer is negative. Indeed, in §**4**, we shall construct a counter-example. $\mathfrak{G}$ and $\mathfrak{H}$ are both countable, torsion-free groups, decomposable into groups of rank $\leqslant 2$.

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[1] A group $\mathfrak{G}$ is complete (or divisible) if each equation $nx = g$, $g \in \mathfrak{G}$ for every natural number n has at least one solution in $\mathfrak{G}$, that is if $n\mathfrak{G} = \mathfrak{G}$.

However, it is proved in **(5)**, using the results of §2, that for the class of completely decomposable groups[2] the answer is positive.

Equivalence by direct summands. In the main the problem is unsolved. The answer is positive for the class of countable periodic groups (Kaplansky **(6)**) and the class of completely decomposable groups **(5)**.

In general, the author expects a negative answer in the case of torsion-free groups. However, the structure of the torsion-free groups seems not yet sufficiently cleared to overcome the difficulties involved. The problem is difficult (Massey **(11)**); for its topological consequences and some other positive results, see Yang **(13)**.

In §3 we construct a simple example of an *indecomposable abelian group of arbitrary finite rank.*

2. Positive results.

THEOREM I.[3] *If $\mathfrak{G}$ and $\mathfrak{H}$ are equivalent by direct summands ($\phi\mathfrak{G}$ and $\psi\mathfrak{H}$ in* (1) *being direct summands) and if in one of the groups the sum of an ascending sequence of direct summands is again a (proper or improper) direct summand, then $\mathfrak{G}$ and $\mathfrak{H}$ are isomorphic.*

Moreover, it is possible to determine direct decompositions

$$\mathfrak{G} = \mathfrak{G}_1 + \mathfrak{G}_2 \tag{2}$$

$$\mathfrak{H} = \mathfrak{H}_1 + \mathfrak{H}_2 \tag{3}$$

such that the relations

$$\phi\mathfrak{G}_1 = \mathfrak{H}_1 \tag{4}$$

$$\psi^{-1}\mathfrak{G}_2 = \mathfrak{H}_2 \tag{5}$$

hold true and define an isomorphic map of $\mathfrak{G}$ onto $\mathfrak{H}$.

Proof. Consider a direct decomposition

$$\mathfrak{G} = \mathfrak{K} + \psi\mathfrak{H}, \tag{6}$$

and define $\mathfrak{G}_1$ by

$$\mathfrak{G}_1 = \mathfrak{K} + \psi\phi\mathfrak{K} + \psi\phi\psi\phi\mathfrak{K} + \dots. \tag{7}$$

This makes sense, since the property of being a direct summand is transitive. $\psi\phi\mathfrak{K}$ is a direct summand of $\psi\mathfrak{H}$, so

$$\mathfrak{G} = \mathfrak{K} + \psi\phi\mathfrak{K} + \mathfrak{X},$$

and so on.

Since the sequence of partial sums in (7) are direct summands of $\mathfrak{G}$, the subgroup $\mathfrak{G}_1$ is a direct summand of $\mathfrak{G}$. $\phi\mathfrak{G}_1$ is a direct summand of $\phi\mathfrak{G}$, $\phi\mathfrak{G}$

[2] A group, decomposable into groups of rank 1, is called completely decomposable. A group is said to be of rank 1, if every finite subset of elements generates a cyclic subgroup (cf. **7**).

[3] This theorem admits considerable generalizations; its proper formulation will be in terms of lattices or semi-lattices.

a direct summand of H, so $\phi\mathfrak{G}_1$ is a direct summand of $\mathfrak{H}$. Applying the map ψ, we see that

$$\psi\phi\mathfrak{G}_1 = \psi\phi\mathfrak{K} + \psi\phi\psi\phi\mathfrak{K} + \dots \tag{8}$$

is a direct summand of $\psi\mathfrak{H}$. Thus there exists a decomposition

$$\psi\mathfrak{H} = \psi\phi\mathfrak{G}_1 + \mathfrak{G}_2, \tag{9}$$

which defines $\mathfrak{G}_2$ uniquely up to isomorphism. Using (6), we see

$$\mathfrak{G} = \mathfrak{K} + \psi\phi\mathfrak{G}_1 + \mathfrak{G}_2.$$

It follows from (7) and (8) that

$$\mathfrak{G}_1 = \mathfrak{K} + \psi\phi\mathfrak{G}_1,$$

so we are in accordance with (2).

Defining $\mathfrak{H}_1$ and $\mathfrak{H}_2$ by (4) and (5), we find, applying ψ^{-1} to (9)

$$\begin{aligned}\mathfrak{H} &= \phi\mathfrak{G}_1 + \psi^{-1}\mathfrak{G}_2 \\ &= \mathfrak{H}_1 + \mathfrak{H}_2,\end{aligned}$$

in accordance with (3).

Thus (4) and (5) define the required isomorphic map of $\mathfrak{G}$ on $\mathfrak{H}$, q.e.d.

We give the following simple application.

COROLLARY. *If $\mathfrak{G}$ and $\mathfrak{H}$ are equivalent complete groups or equivalent additive groups of a division-ring, then $\mathfrak{G}$ and $\mathfrak{H}$ are isomorphic.*

Proof. Each subgroup $\mathfrak{U}$ of an additive group of a division-ring with characteristic $p \neq 0$ is a direct summand, so we can apply Theorem I, if the characteristic equals $p \neq 0$. In case $p = 0$, the additive group of a division-ring is complete, so we have only to consider equivalent complete groups $\mathfrak{G}$ and $\mathfrak{H}$. Since a subgroup of a complete group is a direct summand if and only if it is complete and since the sum of an arbitrary number of complete subgroups is again complete, we can apply Theorem I to obtain the required result.

It has to be noted that this result can also be obtained easily by applying the set-theoretical Schröder-Bernstein theorem directly in view of the (up to isomorphisms) unique decomposition of a complete group into a direct sum of groups of rational numbers and groups of type p^∞ for various prime numbers p.

This corollary can be generalized to more general classes of completely decomposable groups, but these are not very interesting since strong conditions must be imposed to avoid counter-examples like Example (i).

The condition in Theorem I requiring that the sum of an ascending sequence of direct summands is again a direct summand is also a very strong one. It is satisfied, for example, for the class of groups **(4)** in which every serving subgroup is a direct summand, but one can prove that there are already

serving subgroups in a free abelian group of infinite rank which are the sums of an ascending sequence of direct summands, but nevertheless not direct summands themselves.

THEOREM II. *Groups $\mathfrak{G}$ and $\mathfrak{H}$, equivalent by serving subgroups, of which at least one has a base, are isomorphic.*

Proof. Since subgroups of a group with a base have themselves a base, $\mathfrak{G}$ and $\mathfrak{H}$ are both expressible (up to isomorphisms uniquely) as a direct sum of infinite cyclic groups and finite primary cyclic groups. Take decompositions of this kind in $\mathfrak{G}$ and $\mathfrak{H}$.

$\mathfrak{G}$ and $\mathfrak{H}$ have the same rank, since they are equivalent. Hence the number of infinite cyclic summands is the same in both group decompositions. Now take the direct sum $\mathfrak{S}$ of those cyclic summands in $\mathfrak{G}$ which are of order p^k (p and k fixed). We shall show that the number of cyclic summands of this type is the same (in the given decompositions of $\mathfrak{G}$ and $\mathfrak{H}$). $\mathfrak{S}$ is a direct summand of $\mathfrak{G}$, $\phi\mathfrak{G}$ a serving subgroup of $\mathfrak{H}$, $\phi\mathfrak{S}$ therefore a serving subgroup of $\mathfrak{H}$. Using the well-known fact that a periodic serving subgroup with all elements of bounded order is a direct summand, we see that $\phi\mathfrak{S}$ is a direct summand of $\mathfrak{H}$. However, since each decomposition of $\mathfrak{H}$ can be refined to a decomposition into indecomposable cyclic direct summands, and since any two such decompositions are isomorphic, the number of cyclic summands in $\phi\mathfrak{S}$, all of order p^k, is less than or equal to the number of cyclic summands of this type in the decomposition of $\mathfrak{H}$, given above. So the number of cyclic summands of this type in $\mathfrak{G}$ is less than or equal to the corresponding number in $\mathfrak{H}$. Since the converse is equally true, the number is the same.

Now the theorem follows by taking direct sums in $\mathfrak{G}$ and $\mathfrak{H}$ corresponding to different numbers p^k.

3. Indecomposable groups of finite rank. Let α be a transcendental number and put

$$a_i = \alpha^i \qquad (i = 1, 2, \ldots, n).$$

We define the group $\mathfrak{G}$ as the additive group of real numbers

$$\mathfrak{G} = \left\{ \frac{a_1}{p_1^k}, \frac{a_2}{p_2^k}, \ldots, \frac{a_n}{p_n^k}, \frac{a_1 + a_2 + \ldots + a_n}{p^k} \right\},$$

that is, $\mathfrak{G}$ is generated by the elements in the right hand side, where the k are variable integers and $p, p_1, p_2, \ldots, p_n$ are distinct prime numbers.

$\mathfrak{G}$ is an indecomposable group[4] of rank n ($\mathfrak{G}$ is clearly countable and torsion-free).

[4]The first indecomposable groups of rank two were constructed by Levi (10) and Pontrjagin (12, p. 384). Baer (cf. 7, p. 217) proved that every serving subgroup of the additive group of p-adic integers is indecomposable. Erdös (3) gave an example of an indecomposable group of rank two, which is essentially the same as our example for the case $n = 2$, though following a different line of thought.

Proof. Using the transcendency of α we see that $\mathfrak{G}$ only contains elements of $n + 2$ different types, that is, the type 0 (of the infinite cyclic group) and the types of $a_1, a_2, \ldots, a_n$ and

$$b = \sum_{i=1}^{n} a_i .$$

Moreover, we see in the same way that only the elements of the serving subgroups

$$\mathfrak{A}_i = \left\{\frac{a_i}{p_i^{\,k}}\right\} \ (i = 1, 2, \ldots, n) \text{ and } \mathfrak{B} = \left\{\frac{b}{p^k}\right\}$$

have the types of $a_i (i = 1, 2, \ldots, n)$ and b, respectively.

Suppose now that $\mathfrak{G}$ is decomposable

$$\mathfrak{G} = \mathfrak{P} + \mathfrak{Q}.$$

Then each of the groups $\mathfrak{A}_i$ and $\mathfrak{B}$ must be contained in either $\mathfrak{P}$ or $\mathfrak{Q}$ since in the opposite case, as the reader may easily verify, both $\mathfrak{P}$ and $\mathfrak{Q}$ would contain elements of, say, the type of a_i in contradiction of the fact that only the serving subgroup $\mathfrak{A}_i$ of rank 1 contains such elements. Say

$$b \in \mathfrak{P}.$$

The a_i are spread over $\mathfrak{P}$ and $\mathfrak{Q}$, but then they are already contained in $\mathfrak{P}$, since otherwise

$$b = p + q, \qquad q \neq 0,$$

in contradiction to $b \in \mathfrak{P}$.

Hence $\mathfrak{Q} = 0$ and $\mathfrak{G}$ is indecomposable, q.e.d.

4. Example (ii). Define "disjoint" groups $\mathfrak{G}_i (i = 1, 2, \ldots)$

$$\mathfrak{G}_i \simeq \mathfrak{G},$$

where $\mathfrak{G}$ is the group of the preceding section with $n = 2$. Define

$$\mathfrak{G}_0 = \left\{\frac{1}{p_1^{\,n}}\right\},$$

where n is a variable integer. We form the (restricted) direct sums

$$\mathfrak{K} = \sum_{i=0}^{\infty} \mathfrak{G}_i, \quad \mathfrak{K}_1 = \sum_{i=1}^{\infty} \mathfrak{G}_i.$$

It is easy to see, using different transcendental numbers, that $\mathfrak{K}$ can be isomorphically embedded in the additive real group. So $\mathfrak{K}$ is torsion-free and countable.

$\mathfrak{K}$ and its subgroup $\mathfrak{K}_1$ are countable torsion-free groups, which are equivalent by serving subgroups, but $\mathfrak{K}$ and $\mathfrak{K}_1$ are not isomorphic.

Proof. $\mathfrak{K}_1$ is a direct summand, so certainly a serving subgroup of $\mathfrak{K}$. Conversely, we can map $\mathfrak{K}$ isomorphically in $\mathfrak{K}_1$:

$$\mathfrak{K} \to \phi\mathfrak{K} \subset \mathfrak{K}_1,$$

by mapping $\mathfrak{G}_0$ in the natural way on $\mathfrak{H}_1 \subset \mathfrak{G}_1$ (where $\mathfrak{H}_1$ corresponds to the subgroup

$$\left\{\frac{a_1}{p_1^n}\right\}$$

of $\mathfrak{G}$, defined in §3), and $\mathfrak{G}_i$ in the natural way on $\mathfrak{G}_{i+1}$ $(i = 1, 2, \ldots)$. $\phi\mathfrak{K}$ is a serving subgroup of $\mathfrak{K}_1$, since $\mathfrak{K}_1/\phi\mathfrak{K}$ is torsion-free. Thus $\mathfrak{K}$ and $\mathfrak{K}_1$ are equivalent by serving subgroups.

To prove that $\mathfrak{K}$ and $\mathfrak{K}_1$ are not isomorphic, we prove that $\mathfrak{K}_1$ contains no direct summand isomorphic to the direct summand $\mathfrak{G}_0$ of $\mathfrak{K}$.

Suppose, on the contrary, that there is a subgroup

$$\mathfrak{G}_0^* \simeq \left\{\frac{1}{p_1^n}\right\}$$

with

$$\mathfrak{K}_1 = \mathfrak{G}_0^* + \mathfrak{M}. \tag{10}$$

Since (putting $\mathfrak{G}_j = \mathfrak{G}$) the elements a_2/p_2^n are contained in $\mathfrak{G}_j$ for all natural numbers n, and since "unlimited division" in $\mathfrak{G}_0^*$ is only possible by powers of p_1, it follows easily from the decomposition (10) that all a_2/p_1^n are contained in $\mathfrak{M}$. The b/p^n are equally contained in $\mathfrak{M}$. So $b - a_2 = a_1$ and therefore the serving subgroup $\{a_1/p_1^n\}$ is contained in the direct summand $\mathfrak{M}$. Hence

$$\mathfrak{G}_j \subset \mathfrak{M}.$$

This is true for all $j = 1, 2, \ldots$. Therefore

$$\mathfrak{M} = \mathfrak{K}_1$$

in contradiction to (10).

Remark. Actually, we proved the existence of two non-isomorphic groups, the first being isomorphic to a direct summand of the second, and the second being isomorphic to a serving subgroup of the first.

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IMBEDDING CONDITIONS FOR HERMITIAN AND NORMAL MATRICES

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1. Imbedding theorems. Let A, B be two square matrices with complex coefficients, of respective orders n and m, where $n \geq m$. We shall say that B is *imbeddable* in A if there exists a unitary matrix U of order n such that U^*AU contains B as a principal submatrix. In other words, B is said to be imbeddable in A if there exists a matrix V of type $n \times m$ such that $V^*V = I_m$ (= the identity matrix of order m) and $V^*AV = B$.

For Hermitian matrices, the following result holds:

THEOREM 1. *Let A, B be two Hermitian matrices of respective orders n and m, where $n \geq m$. Let $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$ and $\beta_1 \geq \beta_2 \geq \ldots \geq \beta_m$ be the characteristic roots of A and B respectively. Then a necessary and sufficient condition for B to be imbeddable in A is that inequalities*

$$\alpha_i \geq \beta_i, \quad \alpha_{n-i+1} \leq \beta_{m-i+1} \qquad (1 \leq i \leq m) \tag{1}$$

be fulfilled.

The necessity part of Theorem 1 is well known **(2**, p. 75**)**. Inequalities (1) had already been given by Cauchy **(1**, p. 187**)** for real symmetric matrices. To the best of our knowledge, no proof of the sufficiency part has been published except for $m = 1$ or n. Dr. A. J. Hoffman, to whom we described our proof in the summer of 1954, kindly sent us an unpublished proof given by H. Wielandt in 1953 and based on quite different ideas.

For normal matrices, we have the following result:

THEOREM 2. *Let A, B be two normal matrices of respective orders n and $n-1$. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta_1, \beta_2, \ldots, \beta_{n-1}$ be the characteristic roots of A and B respectively. Renumber them so that $\alpha_1, \alpha_2, \ldots, \alpha_q$ are each distinct from $\beta_1, \beta_2, \ldots, \beta_{q-1}$, while $\alpha_j = \beta_{j-1}$ for $q+1 \leq j \leq n$. Then a necessary and sufficient condition for B to be imbeddable in A is that the $2q-1$ points $\alpha_1, \alpha_2, \ldots, \alpha_q, \beta_1, \beta_2, \ldots, \beta_{q-1}$ in the complex plane shall be distinct, collinear, and that every segment on this line limited by two adjacent α_i's $(1 \leq i \leq q)$ shall contain one $\beta_j (1 \leq j \leq q-1)$.*

This result generalizes the case $m = n - 1$ of Theorem 1. It may appear surprising that a generalization to normal matrices is possible. For one thing, the roots are now complex, and since complex numbers have no natural ordering, inequalities (1) might not be extendible.

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We observe that the essential result in Theorem 1 is the case $m = n - 1$, in which case (1) becomes

$$\alpha_1 \geqslant \beta_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_{n-1} \geqslant \beta_{n-1} \geqslant \alpha_n. \tag{2}$$

The result for this case is easily extended to the case $m < n - 1$. In fact, if (1) holds, intermediary sequences of characteristic roots can be inserted such that two consecutive sequences are interlaced similarly to (2). Then, by the result for $m = n - 1$, there exists a chain of Hermitian matrices, with orders increasing by unity, such that each is imbeddable in the next.

For normal matrices, Theorem 2 deals only with the case $m = n - 1$. The case $m < n - 1$ seems to involve additional complications. If a normal matrix B of order $m < n - 1$ is imbeddable in a normal matrix A of order n, it is not true in general that there exists a chain of normal matrices, beginning with B and ending with A, with orders increasing by unity, such that each is imbeddable in the next. As an example, let A, B be diagonal matrices:

$$A = \operatorname{diag}\{0, 1, i, 1 + i\},\ B = \operatorname{diag}\left\{\frac{5 + 8i}{10},\ \frac{5 + 2i}{10}\right\}.$$

If we take

$$V = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 & 2 \\ 1 & -2 \\ 2 & -1 \\ 2 & 1 \end{bmatrix},$$

then $V^*V = I_2$ and $V^*AV = B$; so B is imbeddable in A. By Theorem 2, any normal matrix C of order 3 imbeddable in A must have two of 0, 1, i, $1 + i$ as characteristic roots and its third root on the segment joining the remaining two. But by Theorem 2, B cannot be imbedded in any such matrix C.

Because of interest in extensions of Witt's theorem on quadratic forms in a field, the following corollary is worth noting: If a normal matrix with characteristic roots $\alpha_1, \beta_2, \ldots, \beta_{n-1}$ can be imbedded in one with roots $\alpha_1, \alpha_2, \ldots, \alpha_n$, then the same is true of the respective matrices with roots $\beta_2, \ldots, \beta_{n-1}$ and $\alpha_2, \ldots, \alpha_n$. Another interesting fact is as follows: A necessary and sufficient condition that *all* matrices imbeddable in a normal matrix A be normal is that the characteristic roots of A shall be collinear, or (what is the same thing) that A have the form $w(I + e^{i\theta}H)$ with H Hermitian, w a complex number and θ real. The sufficiency is evident:

$$V^*(I + e^{i\theta}H)V = I_m + e^{i\theta}K,$$

as $K = V^*HV$ is Hermitian. To prove the necessity notice that if N is normal then $w_1N + w_2I$ is normal (w_1 and w_2 complex numbers). Hence it suffices to consider $A = \operatorname{diag}\{0, 1, a + bi\}$, $b \neq 0$. But if

$$V^* = \begin{bmatrix} k & k & k \\ h & -h & 0 \end{bmatrix},\quad h = 1/\sqrt{2},\ k = 1/\sqrt{3},$$

the matrix

$$B = V^*AV = \begin{bmatrix} \frac{1}{2}(1+a+bi) & -hk \\ -hk & \frac{1}{2} \end{bmatrix}$$

is not normal.

2. Proof of the sufficiency part of Theorem 1. According to a remark made above it suffices to consider $m = n - 1$. Since the properties in question are invariant under unitary transformations we need only prove the following: If real numbers $\alpha_i (1 \leqslant i \leqslant n)$ and $\beta_j (1 \leqslant j \leqslant n-1)$ satisfy (2), then there exist $n-1$ complex numbers $z_i (1 \leqslant i \leqslant n-1)$ and a real number γ such that the Hermitian matrix

$$\begin{bmatrix} \gamma & z_1 & z_2 & z_3 & \dots & z_{n-1} \\ \bar{z}_1 & \beta_1 & 0 & 0 & \dots & 0 \\ \bar{z}_2 & 0 & \beta_2 & 0 & \dots & 0 \\ \bar{z}_3 & 0 & 0 & \beta_3 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \dots & \\ \bar{z}_{n-1} & 0 & 0 & 0 & \dots & \beta_{n-1} \end{bmatrix} \tag{3}$$

has $\{\alpha_i\}$ as characteristic roots. To show this, we may assume

$$\alpha_1 > \beta_1 > \alpha_2 > \beta_2 > \dots > \beta_{n-1} > \alpha_n \tag{4}$$

instead of (2). In fact, if $\beta_j = \alpha_j$ or α_{j+1}, we can choose $z_j = 0$ and then work with a matrix of order decreased by unity. Since the characteristic polynomial of the matrix (3) is

$$\left\{\lambda - \gamma - \sum_{j=1}^{n-1} \frac{|z_j|^2}{\lambda - \beta_j}\right\} \cdot \prod_{j=1}^{n-1} (\lambda - \beta_j),$$

and in view of the strict inequalities (4), the requirement that $\{\alpha_i\}$ shall be the characteristic roots of the matrix (3) is equivalent to

$$\gamma + \sum_{j=1}^{n-1} \frac{|z_j|^2}{\alpha_i - \beta_j} = \alpha_i \qquad (1 \leqslant i \leqslant n).$$

Hence it remains to prove the following

LEMMA. *Under the hypothesis* (4), *the system of n linear equations*

$$\frac{x_1}{\alpha_i - \beta_1} + \frac{x_2}{\alpha_i - \beta_2} + \dots + \frac{x_{n-1}}{\alpha_i - \beta_{n-1}} + x_n = \alpha_i \qquad (1 \leqslant i \leqslant n) \tag{5}$$

has a unique solution $x_1, x_2, \dots, x_n$, *and this solution satisfies* $x_k > 0$ $(1 \leqslant k \leqslant n-1)$.

Proof. Let Δ denote the determinant of the coefficient matrix of the system (5). Let Δ_k denote the determinant obtained from Δ by replacing the kth column by the column of quantities α_i. Then we are to prove that $\Delta \neq 0$ and that Δ_k/Δ is positive for $1 \leqslant k \leqslant n-1$.

This will be seen from certain expressions identically equal to Δ or Δ_k. In deriving these expressions we shall regard the α's and β's as variables. First we write

$$\Delta = P/Q, \tag{6}$$

where

$$Q = \prod_{1\leqslant i\leqslant n}\ \prod_{1\leqslant j\leqslant n-1} (\alpha_i - \beta_j), \tag{7}$$

and where P is a polynomial in the $2n-1$ variables α_i and β_j. Since two rows, or two columns, of Δ become equal when two α's, or two β's, are set equal, every difference $\alpha_i - \alpha_j$ $(1 \leqslant i < j \leqslant n)$ and $\beta_i - \beta_j$ $(1 \leqslant i < j \leqslant n-1)$ is a factor of P. The number of these factors being

$$\binom{n}{2} + \binom{n-1}{2} = (n-1)^2,$$

P being homogeneous of degree $(n-1)^2$, we must have

$$P = c \prod_{1\leqslant i<j\leqslant n} (\alpha_i - \alpha_j) \cdot \prod_{1\leqslant i<j\leqslant n-1} (\beta_i - \beta_j),$$

where c is a constant. To determine c we may choose

$$\beta_j = j\ (1 \leqslant j \leqslant n-1),\quad \alpha_i = i + \epsilon\ (1 \leqslant i \leqslant n-1),\quad \alpha_n = 0.$$

After this substitution, if we multiply by ϵ each of the first $n-1$ columns of Δ, then from the complete expansion of the determinant it is easily seen that $\epsilon^{n-1}\Delta \to 1$ as $\epsilon \to 0$. In the polynomial P in ϵ, the constant term is

$$c(n-1)! \prod_{1\leqslant i<j\leqslant n-1} (i-j)^2.$$

Also, we observe that ϵ^{n-1} is a factor of the polynomial Q in ϵ, and that the constant term in the polynomial Q/ϵ^{n-1} is

$$(-1)^{\frac{1}{2}(n-1)n}(n-1)! \prod_{1\leqslant i<j\leqslant n-1} (i-j)^2.$$

So we have $\epsilon^{n-1}\ P/Q \to c(-1)^{\frac{1}{2}(n-1)n}$, as $\epsilon \to 0$. Hence $c = (-1)^{\frac{1}{2}(n-1)n}$ and

$$P = (-1)^{\frac{1}{2}(n-1)n} \prod_{1\leqslant i<j\leqslant n} (\alpha_i - \alpha_j) \cdot \prod_{1\leqslant i<j\leqslant n-1} (\beta_i - \beta_j). \tag{8}$$

Similarly, we write

$$\Delta_k = P_k/Q_k \qquad (1 \leqslant k \leqslant n-1). \tag{9}$$

where

$$Q_k = \prod_{1\leqslant i\leqslant n}\ \prod_{\substack{1\leqslant j\leqslant n-1\\ j\neq k}} (\alpha_i - \beta_j), \tag{10}$$

and

$$P_k = c_k \prod_{1\leqslant i<j\leqslant n} (\alpha_i - \alpha_j) \cdot \prod_{\substack{1\leqslant i<j\leqslant n-1\\ i,\ j\neq k}} (\beta_i - \beta_j).$$

To determine the constant c_k, we use the same substitution as above. If we

multiply by ϵ each column of Δ_k except the kth and nth, then we get $\epsilon^{n-2}\Delta_k \to k$, as $\epsilon \to 0$. One also verifies that

$$\epsilon^{n-2} P_k/Q_k \to kc_k(-1)^{k+\frac{1}{2}n(n-3)}, \qquad \epsilon \to 0.$$

So we have

$$P_k = (-1)^{k+\frac{1}{2}n(n-3)} \prod_{1\leqslant i<j\leqslant n} (\alpha_i - \alpha_j) \cdot \prod_{\substack{1\leqslant i<j\leqslant n-1 \\ i,\ j\neq k}} (\beta_i - \beta_j). \tag{11}$$

From (6), (7), (8) and (4), it is clear that $\Delta \neq 0$. Collecting our results we obtain, after simplification,

$$\Delta_k/\Delta = -\prod_{1\leqslant i\leqslant n} (\alpha_i - \beta_k) \Big/ \prod_{\substack{1\leqslant j\leqslant n-1 \\ j\neq k}} (\beta_j - \beta_k) \qquad (1 \leqslant k \leqslant n-1), \tag{12}$$

which is positive, by (4). This completes the proof.

From the above proof, it is clear that in the case of real symmetric matrices A, B, Theorem 1 remains valid if we require U to be orthogonal in our definition of imbedding.

3. Proof of Theorem 2. We first prove the necessity part. Let A, B be two normal matrices of orders n and $n-1$ respectively. Let α_i $(1 \leqslant i \leqslant n)$ be the characteristic roots of A, and let β_j $(1 \leqslant j \leqslant n-1)$ be those of B. Assume that B is imbeddable in A. Then there exists a unitary matrix U of order n such that U^*AU is of the form

$$U^*AU = \begin{bmatrix} \gamma & z_1 & z_2 & z_3 & \dots & z_{n-1} \\ \bar{w}_1 & \beta_1 & 0 & 0 & \dots & 0 \\ \bar{w}_2 & 0 & \beta_2 & 0 & \dots & 0 \\ \bar{w}_3 & 0 & 0 & \beta_3 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \dots & \\ \bar{w}_{n-1} & 0 & 0 & 0 & \dots & \beta_{n-1} \end{bmatrix}. \tag{13}$$

The fact that U^*AU is normal can be expressed by

$$|z_j| = |w_j| \qquad (1 \leqslant j \leqslant n-1), \tag{14}$$

$$z_j \bar{z}_k = w_j \bar{w}_k \qquad (1 \leqslant j \leqslant n-1,\ 1 \leqslant k \leqslant n-1), \tag{15}$$

$$(\beta_j - \gamma) w_j = \overline{(\beta_j - \gamma)}\, z_j \qquad (1 \leqslant j \leqslant n-1). \tag{16}$$

From (15), we have $(z_j \bar{w}_j)(z_k \bar{z}_k) = (z_k \bar{w}_k)(w_j \bar{w}_j)$, so among the $n-1$ numbers $z_j\bar{w}_j$ $(1 \leqslant j \leqslant n-1)$, the non-zero ones have the same amplitude. Designating this common amplitude by 2θ, we have by (14):

$$z_j = e^{i2\theta} w_j \qquad (1 \leqslant j \leqslant n-1). \tag{17}$$

We may assume that $z_q, z_{q+1}, \dots, z_{n-1}$ are all those z_j's which are zero (if they exist). If $1 \leqslant j \leqslant q-1$, then by (16), either $\beta_j - \gamma = 0$, or

$$\operatorname{amp}(\beta_j - \gamma) \equiv \theta \pmod{\pi}.$$

In either case, we can set

$$\beta_j = \gamma + e^{i\theta} b_j \qquad (1 \leqslant j \leqslant q-1), \tag{18}$$

where b_j is real. Put

$$u_j = e^{-i\theta} x_j \qquad (1 \leqslant j \leqslant q-1). \tag{19}$$

Then by (17),

$$\bar{u}_j = e^{-i\theta} \bar{w}_j \qquad (1 \leqslant j \leqslant q-1). \tag{20}$$

Introduce the Hermitian matrix of order q

$$H = \begin{bmatrix} 0 & u_1 & u_2 & u_3 & \dots & u_{q-1} \\ \bar{u}_1 & b_1 & 0 & 0 & \dots & 0 \\ \bar{u}_2 & 0 & b_2 & 0 & \dots & 0 \\ . & . & . & . & \dots & \\ \bar{u}_{q-1} & 0 & 0 & 0 & \dots & b_{q-1} \end{bmatrix} \tag{21}$$

and let

$$C = \gamma I_q + e^{i\theta} H, \quad D = \operatorname{diag}\{\beta_q, \beta_{q+1}, \dots, \beta_{n-1}\}. \tag{22}$$

Then according to (18), (19) and (20), we have

$$U^* A U = \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}. \tag{23}$$

Let $a_1 \geqslant a_2 \geqslant \dots \geqslant a_q$ be the characteristic roots of the Hermitian matrix H. Then by (22) and (23), the characteristic roots $\{\alpha_i\}$ of A (also of $U^* A U$) can be so renumbered that

$$\begin{cases} \alpha_j = \gamma + e^{i\theta} a_j & (1 \leqslant j \leqslant q), \\ \alpha_j = \beta_{j-1} & (q+1 \leqslant j \leqslant n). \end{cases} \tag{24}$$

Now we renumber $\beta_1, \beta_2, \dots, \beta_{q-1}$ in such a way that the corresponding $b_1, b_2, \dots, b_{q-1}$ (see (18)) are arranged in decreasing order. Then, since $a_1, a_2, \dots, a_q$ are the characteristic roots of H, we have the interlacing inequalities

$$a_1 \geqslant b_1 \geqslant a_2 \geqslant b_2 \geqslant \dots \geqslant b_{q-1} \geqslant a_q. \tag{25}$$

Relations (18), (24) and (25) together express precisely the condition stated in the theorem.

To prove the sufficiency, let complex numbers α_i $(1 \leqslant i \leqslant n)$ and β_j $(1 \leqslant j \leqslant n-1)$ satisfy the condition stated in the theorem. Then for some complex number γ and real numbers

$$\theta;\ a_1, a_2, \dots, a_q;\ b_1, b_2, \dots, b_{q-1},$$

we have the relations (18), (24) and (25). From (25) and Theorem 1, there exists a Hermitian matrix H of order q, with characteristic roots $a_1, a_2, \dots, a_q$, and containing $\operatorname{diag}\{b_1, b_2, \dots, b_{q-1}\}$ as a principal submatrix. Using this H and the numbers γ, θ appearing in (18), (24), we form the normal matrix

$$C = \gamma I_q + e^{i\theta} H.$$

Let $D = \operatorname{diag}\{\beta_q, \beta_{q+1}, \ldots, \beta_{n-1}\}$. Then

$$\begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}$$

is a normal matrix of order n, and by (24), its characteristic roots are $\alpha_1, \alpha_2, \ldots, \alpha_n$. Since H contains $\operatorname{diag}\{b_1, b_2, \ldots, b_{q-1}\}$ as a principal submatrix, so by (18),

$$\begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}$$

contains $\operatorname{diag}\{\beta_1, \beta_2, \ldots, \beta_{n-1}\}$ as a principal submatrix.

This shows that every normal matrix of order $n-1$ with characteristic roots β_j $(1 \leqslant j \leqslant n-1)$ is imbeddable in any normal matrix of order n with characteristic roots α_i $(1 \leqslant i \leqslant n)$.

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INEQUALITIES FOR SYMMETRIC FUNCTIONS AND HERMITIAN MATRICES

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1. Introduction. The purpose of this paper is to present two concavity results for symmetric functions and apply these to obtain inequalities connecting the characteristic roots of the non-negative Hermitian (n.n.h.) matrices A, B and $A + B$.

H. F. Bohnenblust has recently communicated to one of the present authors a suggestion of H. Samelson indicating that Theorem 2 below can be derived from a concavity result on mixed volumes of convex bodies due to Fenchel **(2)**. Bohnenblust has also obtained a separate proof of Theorem 2 by different methods and we are indebted to him for several valuable conversations concerning his results. Theorem 1 in the sequel is believed to be new and we show that it actually provides a simple and direct algebraic proof of Theorem 2 without using the results on mixed volumes. We have not been able to find an explicit statement of Theorem 2 in the literature. Both the present proof and the unpublished proof of Bohnenblust were constructed before Samelson observed the connection with Fenchel's result.

In §2 we state and prove the two results on symmetric functions. In §3 these results are applied to obtain inequalities connecting the characteristic roots of n.n.h. matrices. Theorem 4 extends the classical Minkowski determinant inequality to all of the coefficients in the characteristic polynomial and Theorem 5 extends a recent concavity result of Fan **(1)** for determinants.

2. Results on Symmetric Functions. Let $(a) = (a_1, \ldots, a_n)$ be a set of real numbers and let $E_r(a)$ denote the rth elementary symmetric function of (a), $1 \leqslant r \leqslant n$:

$$E_r(a) = \sum_{1 \leqslant i_1 < \ldots < i_r \leqslant n} \prod_{j=1}^{r} a_{i_j},$$

where the summation extends over all $\binom{n}{r}$ choices of $i_1 < \ldots < i_r$ from $1, \ldots, n$. If $r = 0$ then $E_r(a) = 1$. If $(a) = (a_1, \ldots, a_n)$ and $(b) = (b_1, \ldots, b_n)$ then let $(a + b) = (a_1 + b_1, \ldots, a_n + b_n)$. We confine our attention here to sets (a) of non-negative numbers at least r of whose elements are assumed positive. If there exists a number λ such that

$$a_j = \lambda b_j, \qquad j = 1, \ldots, n,$$

we shall write $(a) \sim (b)$. If this is not the case we indicate it by $(a) \nsim (b)$.

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The set $a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n$ will be denoted by (a'_j). We shall also use the abbreviation

$$\mathbf{E}_r(a) = \frac{E_r(a)}{E_{r-1}(a)}.$$

THEOREM 1. *If* $1 \leqslant r \leqslant n$ *then*

$$\mathbf{E}_r(a+b) > \mathbf{E}_r(a) + \mathbf{E}_r(b)$$

unless $(a) \sim (b)$ *or* $r = 1$ *in which cases we have equality.*

Proof. We use an induction on r. If $(a) \sim (b)$ we clearly have equality, hence assume $(a) \nsim (b)$. For $r = 2$ the result follows from

$$\mathbf{E}_2(a+b) - \mathbf{E}_2(a) - \mathbf{E}_2(b) = \frac{\sum_{i=1}^{n}\left(a_i\sum_{j=1}^{n} b_j - b_i\sum_{j=1}^{n} a_j\right)^2}{2\sum_{i=1}^{n}(a_i+b_i)\sum_{i=1}^{n} a_i\sum_{i=1}^{n} b_i}.$$

Hence assume $r > 2$. We first note that

$$\sum_{i=1}^{n} a_i E_{r-1}(a'_i) = r\,E_r(a). \tag{1}$$

Now

$$E_r(a) = a_i E_{r-1}(a'_i) + E_r(a'_i),$$

and summing on i we have

$$n\,E_r(a) = \sum_{i=1}^{n} a_i E_{r-1}(a'_i) + \sum_{i=1}^{n} E_r(a'_i).$$

Using (1) it follows that

$$\sum_{i=1}^{n} E_r(a'_i) = (n-r)\,E_r(a). \tag{2}$$

Now write

$$\begin{aligned} E_r(a) - E_r(a'_i) &= a_i E_{r-1}(a'_i) \\ &= a_i E_{r-1}(a) - a_i^2 E_{r-2}(a'_i) \end{aligned}$$

summing over i and using (2) we have

$$r\,E_r(a) = \sum_{i=1}^{n} a_i E_{r-1}(a) - \sum_{i=1}^{n} a_i^2 E_{r-2}(a'_i),$$

$$\begin{aligned} \frac{E_r(a)}{E_{r-1}(a)} &= \frac{1}{r}\left\{\sum_{i=1}^{n} a_i - \sum_{i=1}^{n} a_i^2 \frac{E_{r-2}(a'_i)}{E_{r-1}(a)}\right\} \\ &= \frac{1}{r}\left\{\sum_{i=1}^{n} a_i - \sum_{i=1}^{n} \frac{a_i^2}{a_i + \mathbf{E}_{r-1}(a'_i)}\right\}. \end{aligned} \tag{3}$$

From (3) we calculate that

$$\phi = \mathbf{E}_r(a+b) - \mathbf{E}_r(a) - \mathbf{E}_r(b)$$

$$= \frac{1}{r}\sum_{i=1}^{n}\left\{\frac{a_i^2}{a_i + \mathbf{E}_{r-1}(a'_i)} + \frac{b_i^2}{b_i + \mathbf{E}_{r-1}(b'_i)} - \frac{(a_i+b_i)^2}{a_i + b_i + \mathbf{E}_{r-1}(a'_i + b'_i)}\right\}.$$

We make the induction hypothesis that the inequality holds for $r-1$. Since $r \geqslant 3$ we have

$$\mathbf{E}_{r-1}(a'_i + b'_i) > \mathbf{E}_{r-1}(a'_i) + \mathbf{E}_{r-1}(b'_i)$$

unless $(a'_i) \sim (b'_i)$ in which case we have equality. We first consider the case for which there exists an i such that $(a'_i) \nsim (b'_i)$; then it follows that

$$\phi > \frac{1}{r}\sum_{i=1}^{n}\left\{\frac{a_i^2}{a_i + \mathbf{E}_{r-1}(a'_i)} + \frac{b_i^2}{b_i + \mathbf{E}_{r-1}(b'_i)} - \frac{(a_i+b_i)^2}{a_i + b_i + \mathbf{E}_{r-1}(a'_i) + \mathbf{E}_{r-1}(b'_i)}\right\} \tag{4}$$

$$= \frac{1}{r}\sum_{i=1}^{n}\frac{(a_i\,\mathbf{E}_{r-1}(b'_i) - b_i\,\mathbf{E}_{r-1}(a'_i))^2}{(a_i + \mathbf{E}_{r-1}(a'_i))(b_i + \mathbf{E}_{r-1}(b'_i))(a_i + b_i + \mathbf{E}_{r-1}(a'_i) + \mathbf{E}_{r-1}(b'_i))}$$

Now in case $(a'_i) \sim (b'_i)$ for each i then the first inequality in (4) is equality. To resolve this case suppose

$$(a'_i) = \lambda_i\,(b'_i)$$

for each i. Then

$$(a_i\,\mathbf{E}_{r-1}(b'_i) - b_i\,\mathbf{E}_{r-1}(a'_i))^2 = (a_i - \lambda_i b_i)^2\,\omega_i^2$$

where

$$\omega_i = \mathbf{E}_{r-1}(b'_i) \neq 0$$

since at least r of the (b) are positive. But on the other hand $a_i = \lambda_i b_i$ implies $(a) \sim (b)$ contrary to assumption.

Hence in this case the second inequality in (4) is strict and the induction is complete.

THEOREM 2. *If (a) and (b) are sets of n positive variables then for $1 \leqslant r \leqslant n$*

$$E_r^{1/r}(a+b) > E_r^{1/r}(a) + E_r^{1/r}(b) \tag{5}$$

unless $(a) \sim (b)$ or $r = 1$, in which cases we have equality.

To prove (5) we first establish a preliminary

LEMMA. *If $(a'_n) \sim (b'_n)$ and $(a) \nsim (b)$ then the inequality (5) holds.*

Proof.[1] Set

$$ta_i = b_i, \qquad i = 1, \ldots, n-1$$
$$(t+\delta)a_n = b_n, \quad \delta \neq 0.$$

[1]The authors are indebted to K. Hoechsmann for simplifying the original proof of this Lemma.

From

$$E_r(b) = E_r(b'_n) + b_n E_{r-1}(b'_n)$$

we conclude that

$$\begin{aligned} E_r(b) &= t^r E_r(a'_n) + (t+\delta)t^{r-1} a_n E_{r-1}(a'_n) \\ &= t^r E_r(a) + \delta t^{r-1} a_n E_{r-1}(a'_n). \end{aligned}$$

Set

$$g(\delta) = E_r^{1/r}(a+b) - E_r^{1/r}(a) - E_r^{1/r}(b).$$

$$\begin{aligned} g(\delta) = &((t+1)^r E_r(a) + \delta(t+1)^{r-1} a_n E_{r-1}(a'_n))^{1/r} \\ &- E_r^{1/r}(a) - (t^r E_r(a) + \delta t^{r-1} a_n E_{r-1}(a'_n))^{1/r}. \end{aligned}$$

Since $g(0) = 0$ it will suffice to establish

$$\delta g'(\delta) > 0$$

for $\delta \neq 0$, for then g strictly increases from $\delta = 0$. First note that

$$g'(\delta) = \frac{a_n E_{r-1}(a_n)(t+1)^{r-1} E_r^{(r-1)/r}(b) - t^{r-1} E_r^{(r-1)/r}(a+b)}{r(E_r(b) E_r(a+b))^{(r-1)/r}}.$$

The sign of this expression is the same as the sign of

$$\begin{aligned} (t+1)^r E_r(b) - t^r E_r(a+b) &= (t+1)^r t^r E_r(a) + \delta t^{r-1} a_n E_{r-1}(a_n) \\ &\quad - t^r (t+1)^r E_r(a) + \delta(t+1)^{r-1} a_n E_{r-1}(a_n) \\ &= t^{r-1}(t+1)^{r-1} \delta a_n E_{r-1}(a_n). \end{aligned}$$

Hence $g'(\delta)$ has the same sign as δ.

The proof of (5) will be done by a double induction on r and n. The case $r = 2$ and n arbitrary can easily be verified directly. We thus make the induction hypotheses that (5) holds for

(i) $r - 1$ and all $n \geqslant r - 1$,
(ii) r and $n - 1$.

Since we know $r = 2$ and $r = n$ (Hölder's inequality) it will suffice to prove (5) for r, n using (i) and (ii).

By the Lemma we can assume $(a'_n) \sim (b'_n)$ and also $(a) \sim (b)$. Set

$$x^2 = a_n, \quad y^2 = b_n, \quad x = R\cos\phi, \quad y = R\sin\phi.$$

Let $f(x, y) = g(R, \phi)$ be the left side of (5) minus the right side:

$$\begin{aligned} (6) \qquad g(R,\phi) = &(E_{r-1}(a'_n + b'_n)R^2 + E_r(a'_n + b'_n))^{1/r} \\ &- (E_{r-1}(a'_n)R^2\cos^2\phi + E_r(a'_n))^{1/r} \\ &- (E_{r-1}(b'_n)R^2\sin^2\phi + E_r(b'_n))^{1/r}. \end{aligned}$$

We show first that there exists a sufficiently large R_0 such that

$$g(R, \phi) > 0$$

for $0 \leqslant \phi \leqslant 2\pi$ and $R \geqslant R_0$. Set

$$C_j = E_j(a'_n + b'_n),$$
$$A_j = E_j(a'_n),$$
$$B_j = E_j(b'_n).$$

By factoring an R^2 from each bracketed quantity in (6) it is clear that

$$g(R, \phi) > 0$$

if and only if

$$\left(C_{r-1} + \frac{C_r}{R^2}\right)^{1/r} - \left(A_{r-1}\cos^2\phi + \frac{A_r}{R^2}\right)^{1/r} - \left(B_{r-1}\sin^2\phi + \frac{B_r}{R^2}\right)^{1/r} > 0.$$

At $\phi = 0$ and $\phi = 2\pi$ this is clear. By setting

$$h'(\phi) = 0$$

we easily calculate that at a minimum for $h(\phi)$

$$\cos^2\phi = A_{r-1}^{1/(r-1)}(A_{r-1}^{1/(r-1)} + B_{r-1}^{1/(r-1)})^{-1},$$
$$\sin^2\phi = B_{r-1}^{1/(r-1)}(A_{r-1}^{1/(r-1)} + B_{r-1}^{1/(r-1)})^{-1}.$$

Replacing these values in $h(\phi)$ and using (i) we see that

$$h(\phi) > 0$$

and hence

$$f(x, y) = g(R, \phi) > 0$$

for $R \geqslant R_0$. Thus if $f(x, y)$ is to have a non-positive minimum it must occur in the interior of the circle $R = R_0$ where the partials satisfy

$$f_x = 0$$
$$f_y = 0. \tag{7}$$

The equations (7) become

$$x\{((x^2 + y^2)C_{r-1} + C_r)^{(1-r)/r}\ C_{r-1} - (x^2A_{r-1} + A_r)^{(1-r)/r}A_{r-1}\} = 0$$
$$y\{((x^2 + y^2)C_{r-1} + C_r)^{(1-r)/r}\ C_{r-1} - (y^2B_{r-1} + B_r)^{(1-r)/r}\ B_{r-1}\} = 0. \tag{8}$$

If $x = y = 0$ we have the result by (ii). If both the curly bracketed quantities in (8) vanish then the result follows easily from (i) by direct substitution. The less trivial case is, say, $y = 0$ and $x \neq 0$. We must then show

$$f(x, 0) > 0$$

for the value x given by

$$x^2 = \frac{C_{r-1}^{1/(r-1)}\dfrac{A_r}{A_{r-1}} - A_{r-1}^{1/(r-1)}\dfrac{C_r}{C_{r-1}}}{A_{r-1}^{1/(r-1)} - C_{r-1}^{1/(r-1)}}.$$

Replacing this in $f(x, 0)$ we have

$$f(x,0) =$$

$$\left(\frac{C_r C_{r-1}^{1/(r-1)} - C_{r-1}^{r/(r-1)} \dfrac{A_r}{A_{r-1}}}{C_{r-1}^{1/(r-1)} - A_{r-1}^{1/(r-1)}}\right)^{1/r} - \left(\frac{A_{r-1}^{r/(r-1)} \dfrac{C_r}{C_{r-1}} - A_r A_{r-1}^{1/(r-1)}}{C_{r-1}^{1/(r-1)} - A_{r-1}^{1/(r-1)}}\right)^{1/r} - B_r^{1/r}$$

$$= C_{r-1}^{1/(r-1)} \left(\frac{\dfrac{C_r}{C_{r-1}} - \dfrac{A_r}{A_{r-1}}}{C_{r-1}^{1/(r-1)} - A_{r-1}^{1/(r-1)}}\right)^{1/r} - A_{r-1}^{1/(r-1)} \left(\frac{\dfrac{C_r}{C_{r-1}} - \dfrac{A_r}{A_{r-1}}}{C_{r-1}^{1/(r-1)} - A_{r-1}^{1/(r-1)}}\right)^{1/r} - B_r^{1/r}.$$

At this point we use the inequality of Theorem 1. Then

$$f(x,0) > \left(\frac{B_r}{B_{r-1}}\right)^{1/r} (C_{r-1}^{1/(r-1)} - A_{r-1}^{1/(r-1)})^{(r-1)/r} - B_r^{1/r} > 0.$$

The last inequality follows from (i). This completes the induction.

To indicate briefly that the concavity of $E_r^{1/r}(a)$ can be obtained from Fenchel's result we set K_i equal to the segment from 0 to 1 on the ith axis $(i = 1, \ldots, n)$ and we let Q be the unit cube in n-space. For $a = (a_1, \ldots, a_n)$, $a_i \geqslant 0$, form the convex sum

$$K_a = \sum_{i=1}^{n} a_i K_i.$$

For λ, μ non-negative numbers form the volume

$$V(\lambda K_a + \mu Q) = \sum_{j=0}^{n} b_j \lambda^{n-j} \mu^j.$$

The coefficients b_j are the elementary symmetric functions of (a) to within a constant factor and are called the *mixed volumes*. It is easy to check that the concavity result of Fenchel[2] reduces in this case to the statement that $E_r^{1/r}(a)$ is concave.

3. Applications to Matrices. The following result is known **(3; 4)**: *Let*

$$f(x_1, \ldots, x_k) = E_r((Ax_1, x_1), \ldots, (Ax_k, x_k)),$$

where A is a n.n.h. *matrix with characteristic roots $0 \leqslant \alpha_i \leqslant \alpha_{i+1}$ $(i = 1, \ldots, n-1)$. Then*

$$\max f = \binom{k}{r} \left(\sum_{j=1}^{k} \frac{\alpha_{n-j+1}}{k}\right)^r$$

$$\min f = E_r(\alpha_1, \ldots, \alpha_k)$$

where both max and min are taken over all sets of k o.n. *vectors in the unitary n-space.*[3]

[2] A discussion of mixed volumes and a conjecture of this general concavity result may be found in the book by T. Bonnesen and W. Fenchel, *Theorie der konvexen Körper* (Chelsea, New York, 1948), p. 93.

[3] The paper of Ostrowski contains only the value of min f.

THEOREM 3. *Let $A, B, A + B$ be* n.n.h. *with characteristic roots $0 \leqslant \alpha_i \leqslant \alpha_{i+1}$, $0 \leqslant \beta_i \leqslant \beta_{i+1}$, $0 \leqslant \lambda_i \leqslant \lambda_{i+1}$ for $i = 1, \ldots, n-1$, respectively. Then for $1 \leqslant r \leqslant k \leqslant n$*

$$E_r^{1/r}(\lambda_1, \ldots, \lambda_k) \geqslant E_r^{1/r}(\alpha_1, \ldots, \alpha_k) + E_r^{1/r}(\beta_1, \ldots, \beta_k).$$

Proof. Let $x_1, \ldots, x_k$ be an o.n. set of characteristic vectors of $A + B$ such that

$$c_j = ((A + B)x_j, x_j) = \lambda_j \qquad (j = 1, \ldots, k).$$

Set

$$(Ax_j, x_j) = a_j, \quad (Bx_j, x_j) = b_j.$$

Then

$$E_r^{1/r}(\lambda) = E_r^{1/r}(c) = E_r^{1/r}(a + b)$$
$$\geqslant E_r^{1/r}(a) + E_r^{1/r}(b) \geqslant E_r^{1/r}(\alpha) + E_r^{1/r}(\beta).$$

Let

$$x^n + \sum_{j=1}^{n} \rho_j(A)\, x^{n-j}$$

be the characteristic polynomial of A. By setting $k = n$ in Theorem 3 we obtain the following extension of the Minkowski inequality to all the coefficients in the characteristic polynomial:

THEOREM 4. *If A and B are* n.n.h. *then for $1 \leqslant r \leqslant n$*

$$|\rho_r(A + B)|^{1/r} \geqslant |\rho_r(A)|^{1/r} + |\rho_r(B)|^{1/r}.$$

The extension of Fan's result is contained in

THEOREM 5. *If A and B are* n.n.h. *and $\omega + \sigma = 1$, $\omega \geqslant 0$, $\sigma \geqslant 0$, then for $1 \leqslant r \leqslant n$*

$$|\rho_r(\omega A + \sigma B)| \geqslant |\rho_r(A)|^{\omega} |\rho_r(B)|^{\sigma}.$$

Proof. As in the proof of Theorem 3

$$E_r^{1/r}(\lambda) = E_r^{1/r}(\omega(a) + \sigma(b)) \geqslant \omega E_r^{1/r}(a) + \sigma E_r^{1/r}(b)$$
$$\geqslant \omega E_r^{1/r}(\alpha) + \sigma E_r^{1/r}(\beta) \geqslant E_r^{\omega/r}(\alpha)\, E_r^{\sigma/r}(\beta).$$

The result follows by taking rth powers.

THEOREM 6. *Under the same hypotheses as Theorem* 3,

$$k^{1-r}\binom{k}{r-1} \mathbf{E}_r(\lambda_1, \ldots, \lambda_k)$$
$$\geqslant E_r(\alpha_1, \ldots, \alpha_k)\left(\sum_{j=1}^{k} \alpha_{n-j+1}\right)^{1-r} \tag{9}$$
$$+ E_r(\beta_1, \ldots, \beta_k)\left(\sum_{j=1}^{k} \beta_{n-j+1}\right)^{1-r}.$$

Proof. We again choose $x_1, \ldots, x_k$ to be k o.n. characteristic vectors of $A + B$ corresponding respectively to the characteristic roots $\lambda_1, \ldots, \lambda_k$. Then

$$((A + B)x_j, x_j) = \lambda_j, \qquad j = 1, \ldots, k$$

and we have

$$\begin{aligned}\mathbf{E}_r(\lambda_1, \ldots, \lambda_k) &= \mathbf{E}_r(((A + B)x, x)) = \mathbf{E}_r((Ax, x) + (Bx, x)) \\ &> \mathbf{E}_r((Ax, x)) + \mathbf{E}_r((Bx, x)) \\ &> E_r(\alpha)\binom{k}{r-1}^{-1}\left(\sum_{j=1}^{k}\frac{\alpha_{n-j+1}}{k}\right)^{1-r} \\ &+ E_r(\beta)\binom{k}{r-1}^{-1}\left(\sum_{j=1}^{k}\frac{\beta_{n-j+1}}{k}\right)^{1-r},\end{aligned}$$

and (9) follows.

COROLLARY 1. *Under the same hypotheses as Theorem* 4

$$n^{1-r}\binom{n}{r-1}\left|\frac{\rho_r(A + B)}{\rho_{r-1}(A + B)}\right| > \frac{|\rho_r(A)|}{(\operatorname{tr} A)^{r-1}} + \frac{|\rho_r(B)|}{(\operatorname{tr} B)^{r-1}}.$$

Proof. Set $k = n$ in (9). We immediately conclude from Corollary 1 that

$$\det(A + B) > \operatorname{tr}(A + B)\left(n^{-\frac{1}{2}n(n-1)}\prod_{r=2}^{n}\binom{n}{r-1}^{-1}\sigma_r\right)$$

where

$$\sigma_r = \frac{|\rho_r(A)|}{(\operatorname{tr} A)^{r-1}} + \frac{|\rho_r(B)|}{(\operatorname{tr} B)^{r-1}}.$$

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ON THE MAXIMUM PRINCIPLE OF KY FAN

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1. Introduction. In 1951 Fan **(1)** proved the following interesting extreme value result: *Let $A_1, \ldots, A_m$ be completely continuous operators on a Hilbert space $\mathfrak{H}$. For $\sigma = 1, 2, \ldots, m$ let $\lambda_{\sigma i} \geqslant \lambda_{\sigma, i+1}$ be the characteristic roots of $A_\sigma^* A_\sigma$. Then, for any positive integer k,*

$$\max \left| \sum_{i=1}^{k} (U_1 A_1 \ldots U_m A_m x_i, x_i) \right| = \sum_{i=1}^{k} \left(\prod_{\sigma=1}^{m} \lambda_{\sigma i} \right)^{\frac{1}{2}}, \tag{1}$$

$$\max_{i,\, j=1,2,\ldots,k} |\det\{(U_1 A_1 \ldots U_m A_m x_i, x_j)\}|^2 = \prod_{i=1}^{k} \prod_{\sigma=1}^{m} \lambda_{\sigma i}, \tag{2}$$

where both maxima are taken over all unitary operators $U_1, \ldots, U_m$ and all sets of k orthonormal (o.n.) *vectors.*

Fan proved (1) for $m = 1$ and then applied an inequality of Pólya **(7)** and a recent result of Horn **(3)** to obtain the theorem for $m \geqslant 2$. This result is a generalization of a result of von Neumann **(10)**, which states that *if A and B are n-square complex matrices with singular values* **(12)** $\alpha_i \geqslant \alpha_{i+1}$, $\beta_i \geqslant \beta_{i+1}$ $(i = 1, 2, \ldots, n-1)$, *then*

$$\max |\mathrm{tr}(UAVB)| = \sum_{i=1}^{n} \alpha_i \beta_i, \tag{3}$$

where the maximum is taken over all unitary U and V.

In this paper we shall confine our attention to the case of finite matrices. We shall show that both (1) and (2) are special cases of a general maximum result for compound operators (Theorem 3). As applications we obtain inequalities analogous to those of Horn **(3)** and Ostrowski on the singular values of a product. An inequality of S. N. Roy **(8)** (later published with a different proof by B. Sz. Nagy **(9)**) is a special case of Theorem 3 and Ostrowski's inequalities **(6)** connecting Schur-convex functions of singular values and characteristic roots.

2. Fan's first result. Before proceeding, we point out that for finite matrices (1) follows immediately from (3). Following Fan we need only prove (1) in the case $m = 1$.

Let $x_1, \ldots, x_k$ be an o.n. set of vectors, $k \leqslant n$, and let P denote the orthogonal projection into the subspace spanned by $x_1, \ldots, x_k$. Then

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$$\begin{aligned}\left|\sum_{i=1}^{k}(U_1A_1x_i, x_i)\right| &= \left|\sum_{i=1}^{k}(U_1A_1x_i, Px_i)\right| \\ &= \left|\sum_{i=1}^{k}(PU_1A_1x_i, x_i)\right| \\ &= |\mathrm{tr}(PU_1A_1)| \\ &\leqslant \max_{U_1,V}|\mathrm{tr}(VPU_1A_1)| \\ &\leqslant \sum_{i=1}^{n}\rho_i\sqrt{\lambda_i},\end{aligned}$$

by (3), where the ρ_j are the singular values of P. Since $\rho_j = 1$ for $j = 1, \ldots, k$, and $\rho_j = 0$ for $j \geqslant k+1$, (1) follows for $m = 1$.

3. Properties of the Grassmann algebra. Let $x_i (i = 1, \ldots, r)$, $1 \leqslant r \leqslant n$ be vectors in unitary n-space V_n. Then

$$z = x_1 \wedge x_2 \wedge \ldots \wedge x_r \tag{4}$$

denotes the Grassmann exterior product of the x_i; it is a vector in V_m where $m = \binom{n}{r}$. If A is a linear transformation of V_n, the rth induced compound of A is a linear transformation of V_m, defined by

$$C_r(A)\, x_1 \wedge \ldots \wedge x_r = Ax_1 \wedge \ldots \wedge Ax_r. \tag{5}$$

The following properties, which we list for later reference, can be found in **(11**, pp. 63–67**)**:

(i) $(x_1 \wedge \ldots \wedge x_r,\ y_1 \wedge \ldots \wedge y_r) = \det\{(x_i, y_j)\}_{i,j=1,\ldots,r}$.

(ii) $C_r(AB) = C_r(A)\, C_r(B)$, $C_r(A^{-1}) = C_r^{-1}(A)$, $C_r(A^*) = [C_r(A)]^*$, where A^* is the transposed conjugate of A.

(iii) The characteristic roots of $C_r(A)$ are the $\binom{n}{r}$ products of the characteristic roots of A taken r at a time.

(iv) If A has any of the following properties, so does $C_r(A)$: non-singular, normal, Hermitian, non-negative Hermitian (n.n.h.), unitary.

To prune some of the foliage of indices usually necessary in discussing these objects we introduce some notation. The set of $\binom{k}{r}$ distinct sequences of positive integers $i_1, \ldots, i_r$ satisfying $1 \leqslant i_1 < \ldots < i_r \leqslant k$ will be denoted by Q_{kr}, and a typical such sequence will be called ω. If $x_1, \ldots, x_k$ is a set of k vectors in V_n, and ω is the set $\{i_1, \ldots, i_r\}$ in Q_{kr}, we set

$$x_\omega = x_{i_1} \wedge \ldots \wedge x_{i_r}.$$

If (a) is a set of n numbers $a_1, \ldots, a_n$, $E_r(a) = E_r(a_1, \ldots, a_n)$ will denote the rth elementary symmetric function of (a).

4. Results. The following theorem is a generalization of the result in **(5)**:

THEOREM 1. *Let H be* n.n.h. *with characteristic roots $h_j \geqslant h_{j+1}$, $j = 1, \ldots, n-1$. Let $f(t) = t^s$, s real, and*

$$\phi(x_1, \ldots, x_k) = \sum_{\omega \in Q_{kr}} f[(C_r(H)x_\omega, x_\omega)],$$

where $1 \leqslant r \leqslant k \leqslant n$. *Then, if* $s \geqslant 1$,

$$\max \phi = E_r[f(h_1), \ldots, f(h_k)]; \tag{6}$$

and if $0 \leqslant s \leqslant 1$,

$$\min \phi = E_r[f(h_{n-k+1}), \ldots, f(h_n)]; \tag{7}$$

where the maximum and minimum are taken over all sets of k o.n. *vectors* $x_1, \ldots, x_k$ *in* V_n.

Proof. For the case $s = 1$, the theorem is proved in **(5)**. The present proof is an application of this result to $f(H)$.

First note that[1] if $||x|| = 1$,

$$f[(Hx, x)] \leqslant (f(H)x, x). \tag{8}$$

For, if $x_1, \ldots, x_n$ are o.n. characteristic vectors of H corresponding to $h_1, \ldots, h_n$, then

$$\begin{aligned} f[(Hx, x)] &= f\left[\sum_{j=1}^{n} h_j |(x, x_j)|^2\right] \\ &\leqslant \sum_{j=1}^{n} f(h_j) |(x, x_j)|^2 \\ &= (f(H)x, x). \end{aligned}$$

Secondly,[2]

$$f(C_r(H)) = C_r(f(H)). \tag{9}$$

For, if U is a unitary matrix such that $U^*HU = D$, diagonal,

$$\begin{aligned} f(C_r(H)) &= f(C_r(UDU^*)) = C_r(U) f(C_r(D)) C_r(U^*) \\ &= C_r(U) C_r(f(D)) C_r(U^*) = C_r(f(UDU^*)) = C_r(f(H)). \end{aligned}$$

To establish (6) we observe that by (8), (9) and the remark at the beginning of the proof,

$$\begin{aligned} \phi &\leqslant \sum_{\omega \in Q_{kr}} (f(C_r(H))x_\omega, x_\omega) = \sum_{\omega \in Q_{kr}} (C_r(f(H))x_\omega, x_\omega) \\ &\leqslant E_r(f(h_1), \ldots, f(h_k)). \end{aligned}$$

Equality can be achieved by choosing $x_1, \ldots, x_k$ to be an o.n. set of characteristic vectors of H corresponding to $h_1, \ldots, h_k$, since by (i),

$$\begin{aligned} \phi &= \sum_{\omega \in Q_{kr}} f(\det\{(Hx_{i_s}, x_{i_t})\}), \qquad s, t = 1, \ldots, r, \\ &= \sum_{\omega \in Q_{kr}} f(\det\{h_{i_s} \delta_{i_s i_t}\}) = \sum_{\omega \in Q_{kr}} f\left(\prod_{s=1}^{r} h_{i_s}\right) \\ &= E_r(f(h_1), \ldots, f(h_k)). \end{aligned}$$

(7) is proved similarly.

[1] This inequality holds for any continuous convex function f defined on the spectrum of H. The inequality is reversed if f is concave.

[2] This equality holds for any function f defined on the spectrum of H and satisfying the relation $f(xy) = f(x) f(y)$.

Theorem 3 is our main result. Theorem 2 is a special case. However, it is the first step in the proof of the general theorem, and its proof appears to be of some interest in itself.

THEOREM 2. *Let A be an arbitrary n-square complex matrix with singular values $\alpha_i \geqslant \alpha_{i+1} (i = 1, \ldots, n-1)$. Let*

$$\phi(x_1, \ldots, x_k; U) = \left| \sum_{\omega \in Q_{kr}} (C_r(UA)x_\omega, x_\omega) \right|,$$

where $1 \leqslant r \leqslant k \leqslant n$. Then

$$\max \phi = E_r(\alpha_1, \ldots, \alpha_k), \tag{10}$$

where the maximum is taken over all sets of k o.n. vectors $x_1, \ldots, x_k$ and all unitary U.

Proof. Since $A = VH$, where V is unitary and $H = (A^*A)^{\frac{1}{2}}$ is n.n.h., we may without loss in generality replace A by H. Set $y_j = U^{-1}x_j$, $(j = 1, \ldots, k)$. By the notation $\sum \oplus C_r(A)$ we mean the direct sum of $C_r(A)$ taken $\binom{k}{r}$ times; and similarly for $\sum \oplus x_\omega$. Since $\sum \oplus C_r(H)$ is n.n.h. by (iv),

$$\begin{aligned}
\phi &= \left| \sum_{\omega \in Q_{kr}} (\iota_r(H)x_\omega, C_r(\iota^{-1})x_\omega) \right| \\
&= \left| \left(\sum \oplus C_r(H) \sum \oplus x_\omega, \sum \oplus y_\omega \right) \right| \\
&\leqslant \left(\sum \oplus C_r(H) \sum \oplus x_\omega, \sum \oplus x_\omega \right)^{\frac{1}{2}} \\
&\qquad \left(\sum \oplus C_r(H) \sum \oplus y_\omega, \sum \oplus y_\omega \right)^{\frac{1}{2}} \\
&= \left[\sum_{\omega \in Q_{kr}} (C_r(H)x_\omega, x_\omega) \right]^{\frac{1}{2}} \left[\sum_{\omega \in Q_{kr}} (C_r(H)y_\omega, y_\omega) \right]^{\frac{1}{2}} \\
&\leqslant E_r(\alpha_1, \ldots, \alpha_k).
\end{aligned}$$

The last inequality follows from (6) applied to each of the square roots. Equality holds when the x_j are suitable eigenvectors of H and U is the identity matrix.

THEOREM 3. *Let $A_1, \ldots, A_m$ be arbitrary n-square complex matrices with singular values $\alpha_{\sigma i}$, where $\alpha_{\sigma i} \geqslant \alpha_{\sigma, i+1}$ $(i = 1, \ldots, n-1;\ \sigma = 1, \ldots, m)$. Let*

$$\phi(x_1, \ldots, x_k; U_1, \ldots, U_m) = \left| \sum_{\omega \in Q_{kr}} (C_r(U_1A_1 \ldots U_mA_m)x_\omega, x_\omega) \right|,$$

where $1 \leqslant r \leqslant k \leqslant n$. Then

$$\max \phi = E_r\left(\prod_{\sigma=1}^{m} \alpha_{\sigma 1}, \ldots, \prod_{\sigma=1}^{m} \alpha_{\sigma k} \right), \tag{11}$$

where the maximum is taken over all sets of k o.n. vectors $x_1, \ldots, x_k$, and all unitary matrices $U_1, \ldots, U_m$.

Proof. For a fixed $U_2, \ldots, U_m$, let $\beta_1 \geqslant \beta_2 \geqslant \ldots \geqslant \beta_n$ be the singular values of $B = A_1(U_2A_2) \ldots (U_mA_m)$. By Theorem 2

$$\phi \leqslant E_r(\beta_1, \ldots, \beta_k). \tag{12}$$

The singular values of $U_\sigma A_\sigma$ are the $\alpha_{\sigma i}$ $(i = 1, \ldots, n)$. By a result of Ostrowski (**6**, p. 283, equation 106a),

$$E_r(\beta_1, \ldots, \beta_k) \leqslant E_r\left(\prod_{\sigma=1}^{m} \alpha_{\sigma 1}, \ldots, \prod_{\sigma=1}^{m} \alpha_{\sigma k}\right).$$

There remains to show that the right side of (11) is taken on for a suitable choice of the x's and U's. As in Theorem 2 we may replace each A_σ by $H_\sigma = (A_\sigma^* A_\sigma)^{\frac{1}{2}}$. Let $x_{\sigma 1}, \ldots, x_{\sigma n}$ $(\sigma = 1, \ldots, m)$ be an o.n. set of characteristic vectors of H_σ corresponding to $\alpha_{\sigma 1}, \ldots, \alpha_{\sigma n}$. Choose the U_σ so that

$$\begin{aligned} U_\sigma x_{\sigma i} &= x_{\sigma-1,i}, \qquad \sigma = 2, \ldots, m, \\ U_1 x_{1i} &= x_{mi}, \end{aligned}$$

for $i = 1, \ldots, n$. Set

$$x_{\sigma\omega} = x_{\sigma i_1} \wedge \ldots \wedge x_{\sigma i_r}$$

and

$$\alpha_{\sigma\omega} = \prod_{j=1}^{r} \alpha_{\sigma i_j}$$

where ω is the set $\{i_1, \ldots, i_r\}$. Then

$$\begin{aligned} &\left| \sum_{\omega \in Q_{kr}} (C_r(U_1H_1 \ldots U_mH_m)\, x_{m\omega}, x_{m\omega}) \right| \\ &= \left| \sum \alpha_{m\omega}\, C_r(U_1H_1 \ldots U_{m-1}H_{m-1})\, x_{m-1,\omega}, x_{m\omega}) \right| \\ &= \sum \alpha_{m\omega} \ldots \alpha_{1\omega}(x_{m\omega}, x_{m\omega}) \\ &= \angle\left(\prod_{\sigma=1}^{m} \alpha_{\sigma i_1} \ldots \prod_{\sigma=1}^{m} \alpha_{\sigma i_r}\right) \\ &= E_r\left(\prod_{\sigma=1}^{m} \alpha_{\sigma 1}, \ldots, \prod_{\sigma=1}^{m} \alpha_{\sigma k}\right). \end{aligned}$$

This completes the proof of the theorem.

Remarks. I. By setting $r = 1$ in (11) we obtain (1); by setting $r = k$ we obtain (2).

II. Theorem 3 does not follow immediately from (1) applied to the compound because the lexicographic ordering of the eigenvalues does not necessarily correspond to the ordering by magnitude.

III. Let A_σ, $\sigma = 1, \ldots, m$, have eigenvalues $\lambda_{\sigma i}$, ordered so that $|\lambda_{\sigma i}| \geqslant |\lambda_{\sigma,i+1}|$ $(i = 1, \ldots, n-1)$. We can find unitary V_σ such that $V_\sigma^* A_\sigma V_\sigma = T_\sigma$, triangular, with the diagonal elements ordered by absolute

magnitude. Then if $U_\sigma = V_{\sigma-1} V_\sigma^*$ $(\sigma = 2, \ldots, m)$, $U_1 = V_m U V_1^*$ where U is the diagonal matrix with

$$\exp\left(-\sum_{\sigma=1}^{m} \arg \lambda_{\sigma i}\right)$$

in the ith row and ith column; and if $x_j = V_m e_j$ $(j = 1, \ldots, k)$, where e_j is the unit vector with 1 in the jth position; then

$$\left|\sum_{\omega \in Q_{kr}} (C_r(U_1 A_1 \ldots U_m A_m)x_\omega, x_\omega)\right| = E_r\left(\prod_{\sigma=1}^{m} |\lambda_{\sigma 1}|, \ldots, \prod_{\sigma=1}^{m} |\lambda_{\sigma k}|\right).$$

It follows from Theorem 3 that

$$E_r\left(\prod_{\sigma=1}^{m} |\lambda_{\sigma 1}|, \ldots, \prod_{\sigma=1}^{m} |\lambda_{\sigma k}|\right) \leqslant E_r\left(\prod_{\sigma=1}^{m} \alpha_{\sigma 1}, \ldots, \prod_{\sigma=1}^{m} \alpha_{\sigma k}\right). \tag{13}$$

The case $m = 1$, $r = k$ is Weyl's inequality. Actually, (13) also follows easily from Ostrowski's discussion of Schur-convex functions.

5. Applications. Let $X = AB$, where A and B are arbitrary n-square complex matrices. Consider the convex sum

$$Y = \sigma X^* X + \delta X X^*,$$

where $\sigma + \delta = 1$, $\sigma \geqslant 0$, $\delta \geqslant 0$. Let $\phi_i \geqslant \phi_{i+1}$, $\alpha_i \geqslant \alpha_{i+1}$, $\beta_i \geqslant \beta_{i+1}$ $(i = 1, \ldots, n-1)$ be respectively the non-negative square roots of the characteristic roots of Y and the singular values of A and B. In the Theorem 4 we shall use the following concavity property of the elementary symmetric functions:[3]

If (a) and (b) are sets of n non-negative numbers and $1 \leqslant r \leqslant n$, then

$$E_r^{1/r}(a + b) \geqslant E_r^{1/r}(a) + E_r^{1/r}(b).$$

THEOREM 4. *For $0 \leqslant s \leqslant 1$ and $1 \leqslant r \leqslant k \leqslant n$,*

$$E_r(\phi_n^{2s}, \ldots, \phi_{n-k+1}^{2s})$$
$$\geqslant (2\sigma^\sigma \delta^\delta)^{r(s-1)}\left(\prod_{j=1}^{r} \alpha_{n-j+1}^{\sigma} \beta_{n-j+1}^{\delta}\right)^{2s} E_r^{\sigma}(\beta_n^{2s}, \ldots, \beta_{n-k+1}^{2s})\, E_r^{\delta}(\alpha_n^{2s}, \ldots, \alpha_{n-k+1}^{2s}).$$

Proof. Let $x_1, \ldots, x_k$ be an o.n. set of characteristic vectors of Y corresponding respectively to

$$\phi_n^2, \ldots, \phi_{n-k+1}^2.$$

Then

[3] This property follows from a concavity property proved by W. Fenchel **(2)**. This inequality also follows directly as a corollary of a similar property for symmetric functions proved by Lopes and Marcus in **(4)**.

$$
\begin{aligned}
E_r^{1/r}(\phi_n^{2s}, \ldots, \phi_{n-k+1}^{2s}) &= E_r^{1/r}[(Yx_j, x_j)^s] \\
&= E_r^{1/r}[\{\sigma(X^*X\,x_j, x_j) + \delta(XX^*x_j, x_j)\}^s] \\
&\geqslant E_r^{1/r}[2^{s-1}\{\sigma^s(X^*X\,x_j, x_j)^s + \delta^s(XX^*\,x_j, x_j)^s\}] \\
&\geqslant 2^{s-1}\{E_r^{1/r}[\sigma^s(X^*X\,x_j, x_j)^s] + E_r^{1/r}[\delta^s(XX^*x_j, x_j)^s]\} \\
&= 2^{s-1}\{\sigma^s\, E_r^{1/r}[(X^*X\,x_j, x_j)^s] + \delta^s\, E_r^{1/r}[(XX^*\,x_j, x_j)^s]\}.
\end{aligned}
$$

It follows from the Hadamard determinant theorem that

$$
\begin{aligned}
E_r[(X^*X\,x_j, x_j)^s] &= \sum_{1\leqslant i_1<\ldots<i_r\leqslant k}\left(\prod_{t=1}^{r}(X^*X\,x_{i_t}, x_{i_t})\right)^s \\
&\geqslant \sum_{\omega\in Q_{k,r}}(C_r(X^*X)\,x_\omega, x_\omega)^s.
\end{aligned}
$$

Thus

$$
\begin{aligned}
E_r^{1/r}(\phi_n^{2s}, \ldots, \phi_{n-k+1}^{2s}) &\geqslant 2^{s-1}\,\sigma^s\left\{\sum (C_r(X^*X)\,x_\omega, x_\omega)^s\right\}^{1/r} \\
&\quad + 2^{s-1}\,\delta^s\left\{\sum (C_r(XX^*)\,x_\omega, x_\omega)^s\right\}^{1/r} \\
&= 2^{s-1}\,\sigma^s\left\{\sum (C_r(A^*A)\,C_r(B)\,x_\omega, C_r(B)\,x_\omega)^s\right\}^{1/r} \\
&+ 2^{s-1}\,\delta^s\left\{\sum (C_r(BB^*)\,C_r(A^*)\,x_\omega, C_r(A^*)\,x_\omega)^s\right\}^{1/r} \\
&\geqslant 2^{s-1}\,\sigma^s\left(\prod_{j=1}^{r}\alpha_{n-j+1}^s\right)^{2/r}\left\{\sum (C_r(B^*B)\,x_\omega, x_\omega)^s\right\}^{1/r} \\
&+ 2^{s-1}\,\delta^s\left(\prod_{j=1}^{r}\beta_{n-j+1}^s\right)^{2/r}\left\{\sum (C_r(AA^*)\,x_\omega, x_\omega)^s\right\}^{1/r} \\
&\geqslant \sigma\left\{2^{s-1}\,\sigma^{s-1}\left(\prod_{j=1}^{r}\alpha_{n-j+1}^s\right)^{2/r} E_r^{1/r}(\beta_n^{2s}, \ldots, \beta_{n-k+1}^{2s})\right\} \\
&+ \delta\left\{2^{s-1}\,\delta^{s-1}\left(\prod_{j=1}^{r}\beta_{n-j+1}^s\right)^{2/r} E_r^{1/r}(\alpha_n^{2s}, \ldots, \alpha_{n-k+1}^{2s})\right\}.
\end{aligned}
$$

The last inequality follows by Theorem 1. The result follows by a classical inequality and by taking rth powers of both sides.

We remark that when σ is zero, σ^r is to be taken as 1.

Theorem 4 may be used to relate the characteristic polynomials of the matrices involved. For example,

$$
|p_r((ABB^*A^*)^s)| \geqslant 2^{r(s-1)}\prod_{j=1}^{r}\alpha_{n-j+1}^{2s}\,|p_r((B^*B)^s)|,
$$

where

$$
x^n + \sum_{r=1}^{n} p_r(M)\,x^{n-r}
$$

is the characteristic polynomial of the matrix M.

In **(9)** B. Sz. Nagy proves that if A and B are complex n-square matrices, then

$$\alpha_n \beta_n \leqslant |\lambda_i| \leqslant \alpha_1 \beta_1, \tag{14}$$

where $\alpha_i \geqslant \alpha_{i+1}$, $\beta_i \geqslant \beta_{i+1}$ are the singular values of A and B, and λ_i, $|\lambda_i| \geqslant |\lambda_{i+1}|$, are the characteristic roots of AB. By Theorem 3,

$$E_r(|\lambda_1|, \ldots, |\lambda_k|) \leqslant E_r(\alpha_1 \beta_1, \ldots, \alpha_k \beta_k), \tag{15}$$

for $1 \leqslant r \leqslant k \leqslant n$. By setting $r = k = 1$, we obtain the upper inequality of (14). If $\alpha_n \beta_n = 0$, the lower inequality is trivial. Otherwise A and B are non-singular. The characteristic roots of $(AB)^{-1}$ are the λ_i^{-1}, and the singular values of A^{-1} and B^{-1} are the α_i^{-1} and β_i^{-1}. Hence

$$E_r(|\lambda_n|^{-1}, \ldots, |\lambda_{n-k+1}|^{-1}) \leqslant E_r(\alpha_n^{-1} \beta_n^{-1}, \ldots, \alpha_{n-k+1}^{-1} \beta_{n-k+1}^{-1}). \tag{16}$$

Again setting $r = k = 1$, we obtain the lower inequality of (14).

Both (15) and (16) have immediate generalizations in two directions: first, to a product of more than two matrices, and second, to the more general class of Schur-convex functions.

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